Every smooth cubic 4-fold in $\mathcal{C}_{14}$ is rational

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Joint work with Michele BOLOGNESI from "Some loci of rational cubic fourfolds", arXiv: 1504.05863

$X \subset \mathbb{P}^5_{\mathbb{C}}$ smooth cubic 4-fold

φ = moduli space cubic 4-folds in $\mathbb{P}^5_{\mathbb{C}}$

$\Rightarrow$ (i) φ irreducible quasi-proj. var.

(ii) $\dim \varphi = \dim \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^5}(3))) - \dim \operatorname{Aut} \mathbb{P}^5$

$= 55 - 35$

$= 20$

Let $\Pi = \mathbb{P}^2 \subset \mathbb{P}^5_{\mathbb{C}}$ be a plane

$$\Psi_\delta = \{ X \in \Psi : X \supseteq \Pi \} \subset \Psi$$

$\Longrightarrow$ (i) Ψ_δ closed and irreducible in Ψ

Let $\Pi = \mathbb{P}^2 \subset \mathbb{P}^5_{\mathbb{C}}$ be a <u>plane</u>

$$\varphi_8 = \{ [X] \in \varphi \ : \ X \supseteq \Pi \} \subset \varphi$$

$\Longrightarrow$ (i) φ_8 irreducible and closed in φ

$\dim \mathbb{G}(2,5) = 9$

$X \supseteq \Pi$ 10 conditions

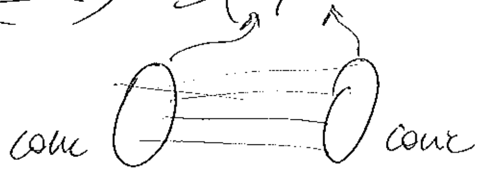
$$\Longrightarrow \operatorname{codim}(\varphi_8, \varphi) = 10 - 9 = 1$$

$\Longrightarrow$ (ii) φ_8 is a <u>divisor</u> in φ

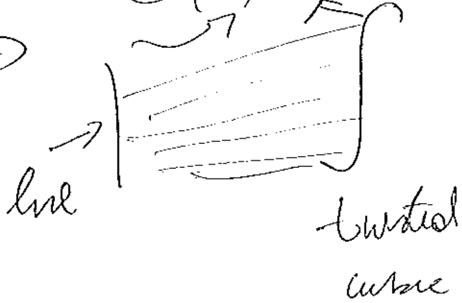
$T \subset \mathbb{P}^5$

smooth rational normal scroll of degree 4

(surface $\simeq$ $\mathbb{P}^1 \times \mathbb{P}^1 \longleftrightarrow S(2,2)$



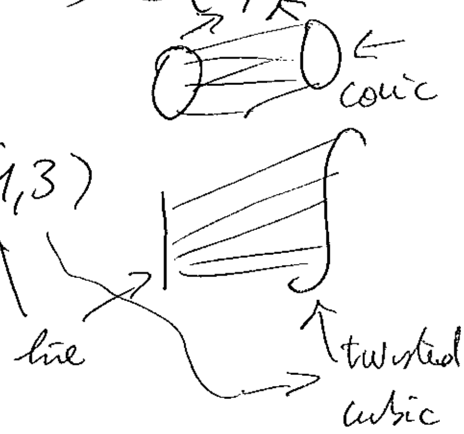
$\mathbb{F}_2 \longleftrightarrow S(1,3)$



$T \subset \mathbb{P}^5$

smooth rational normal scroll of degree 4:

(surface $\simeq \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 \longleftrightarrow S(2,2) \\ \mathbb{F}_2 \longleftrightarrow S(1,3) \end{cases}$



$S \subset \mathbb{P}^5$ smooth quintic

del Pezzo surface

$S \subset \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ as

a divisor of type $(1,2)$

$X \supset S$ general cubic hyp.
containing S

smooth

$$\Rightarrow \underbrace{X \cap (\mathbb{P}^1 \times \mathbb{P}^2)}_{(3,3) \text{ divisor}} = \underset{(1,2)}{S} \cup \underset{(2,1)}{T}$$

$X \supset S$ general cubic hyp containing S

$X \cap (\mathbb{P}^1 \times \mathbb{P}^2) = S \cup T$

$(3,3)$ $(1,2)$ $(2,1)$

$$\mathcal{C}_{14} = \overline{\{[X] \in \mathcal{C} : X \supset T\}}$$

$$= \overline{\{\underbrace{[X] \ni \mathcal{C} : X \supset S}\}}$$

$\overset{\shortparallel}{Pf}$ Pfaffian locus in $\mathcal{C}_{14}$

$\overline{Pf} = \mathcal{C}_{14}$

Why $\mathcal{C}_{14}$ or why $\mathcal{C}_8$?

h class of a hyp. section of X

$$A(X) = A^2(X) \cong \frac{\{2\text{-cycles}\}}{\sim_{alg}}$$

$$\langle h^2, T \rangle = \langle h^2, S \rangle \quad \text{if } X \in \mathcal{C}_{14} \text{ general}$$

$$\begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 4 \\ T & 4 & 10 \end{array} \quad T^2=10 \qquad \begin{array}{c|cc} & h^2 & S \\ \hline h^2 & 3 & 5 \\ S & 5 & 13 \end{array} \quad S^2=13$$

$$\left|\ \ \right| = 14 \qquad \mathcal{C}_{14}$$

$$\langle h^2, \Pi \rangle \qquad \begin{array}{c|cc} & h^2 & \Pi \\ \hline h^2 & 3 & 1 \\ \Pi & 1 & 3 \end{array} \qquad \left|\ \ \right| = 8$$

$$\Pi^2 = 3 \qquad\qquad \mathcal{C}_8$$

<u>Def</u> (OADP variety) W^n equidimensional

$W^n \subset \mathbb{P}^{2n+1}$ is an <u>OADP variety</u>

if through a general $p \in \mathbb{P}^{2n+1}$

there passes <u>a unique secant</u>

<u>line $l_p = \langle p_1, p_2 \rangle$ to W</u>

with $p_i \in W$, $p_1 \neq p_2$

<u>Examples</u> 0) $W \subset \mathbb{P}^3$ twisted cubic

0') $W = \mathbb{P}^1_1 \amalg \mathbb{P}^1_2 \subset \mathbb{P}^3$ skew lines

1) $W = T \subset \mathbb{P}^5$

2) $W = S \subset \mathbb{P}^5$

3) $W = \mathbb{P}^2_1 \amalg \mathbb{P}^2_2 \subset \mathbb{P}^5$ skew planes

<u>Proposition</u> $X \subset \mathbb{P}^{2n+1}$ cubic hypersurface

$X \supset W^n$ OADP-variety

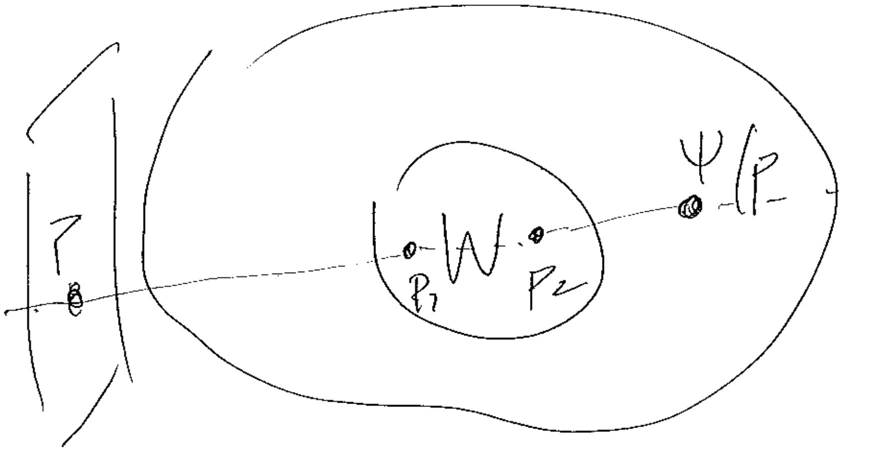
$\Longrightarrow X$ rational

<u>Pf</u> $H = \mathbb{P}^{2n}$ general hyperplane

$p \in H$ general $\quad l_p = \langle p_1, p_2 \rangle$
$\quad p_i \in W$

$\psi : H \dashrightarrow X$

$p \longmapsto (l_p \cap X) \setminus \{p_1, p_2\}$



Zariski Main Theorem
$\Rightarrow$ <u>through</u> <u>a</u> general
$q \in X$ there passes a
unique secant line to
W

Consequence (i) A general

$X \in \mathcal{C}_{14}$ is rational

Rmk a) ~~In~~ dimension ≥ 4
rationality is not known to
be a closed property on the base
of a flat deformation family of
smooth varieties.

b) In dimension 3 it is true
(de Fernex - Fusi) (≤ 2 well known)

<u>Theorem 1</u> (Bolognesi, — ; 2015/16)

Every $X \in \mathcal{C}_{14}$ is rational

<u>Expectation</u>: (i) $X' \in \mathcal{C}_{14}$

$$X_{\lambda_0} = X' \longleftarrow X_\lambda \quad \text{flat proj def}$$

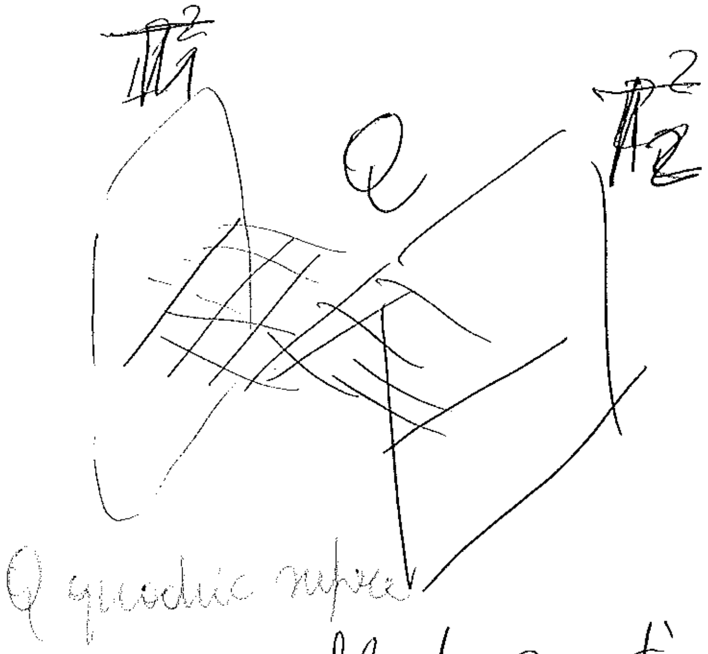
$$\forall \lambda \neq \lambda_0 \quad X_\lambda \supset T = T_\lambda$$

$$\Longrightarrow X'_{\lambda_0} \supset T_{\lambda_0} \quad \text{flat proj def of } T = T_\lambda,\ \lambda \neq \lambda_0$$

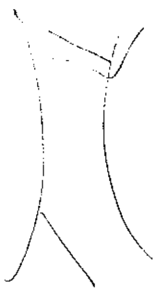
<u>Key point</u>: T_{λ_0} OADP surface, possibly REDUCIBLE (NOT OBVIOUS AT ALL!!)

Example:

$X \supset$



$\mathbb{P}^2_1 \cup \mathbb{P}^2_2 \cup Q$ flat projective deformation of T



$\Longrightarrow X \in \Psi_8 \cap \Psi_{14}$

Remark 1) $X \notin Pf$, i.e. $X \not\supset S$ smooth quintic del Pezzo

2) $X \not\supset T$ smooth rat. scroll of degree 4

(Tregub)

3) $\hat{\Psi}_{17} = \{ [X] \in \Psi_8 : X \supset \Pi_1 \sqcup \Pi_2 \}$ irreducible component of $\Psi_8 \cap \Psi_{14} \Longrightarrow$ Pf [illegible]

Key Remark: If $W_{\lambda_0}^n \subset \mathbb{R}^{2n+1}$ is the flat limit of a family $\{W_\lambda\}_{\lambda \neq \lambda_0}$ of OADP varieties

$\Longrightarrow$ W_{λ_0} is a non-degenerate scheme

Pf $p \in \mathbb{P}^{2n+1}$ general fixed

$\{\ell_x\}$ unique secant line to W_X through p.

Pf $p \in \mathbb{P}^{2n+1}$ generale & fixed

$\{\ell_\lambda\}_{\lambda \neq \lambda_0}$ unique secant

line to W_λ through p.

$[\ell_\lambda]_{\lambda \neq \lambda_0} \in G(1, 2n+1) \Rightarrow \exists$ limit line ℓ_{λ_0}

and $\text{length}(\ell_{\lambda_0} \cap W_{\lambda_0}) \geq 2$

Pf $p \in \mathbb{P}^{2n+1}$ general fixed

$\{l_\lambda\}_{\lambda \neq \lambda_0}$ unique secant line to W_λ through p.

$[l_\lambda]_{\lambda \neq \lambda_0} \in G(1, 2n+1) \Rightarrow \exists$ limit line l_{λ_0} passing through p

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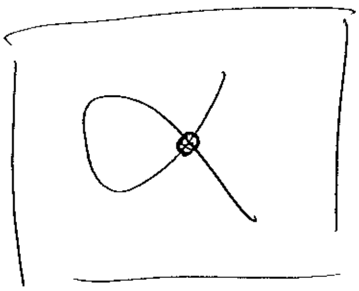
$\Rightarrow \text{length}(l_{\lambda_0} \cap W_{\lambda_0}) \geq 2$

$\Rightarrow p \in \langle W_{\lambda_0} \rangle \Rightarrow \langle W_{\lambda_0} \rangle = \mathbb{P}^{2n+1}$ $\square$

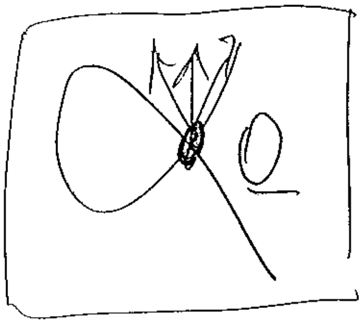
$\underline{\text{Ex}}$ $W_\lambda \subset \mathbb{P}^3$ family of twisted cubics = OADP curves

$\lambda \neq \lambda_0$

$W_\lambda \longrightarrow W_{\lambda_0} =$ 

$(W_{\lambda_0})_{red}$ = plane cubic with a node

<u>Ex</u> $W_\lambda \subset \mathbb{P}^3$ family of twisted cubics = OADP irreducible curves

$W_\lambda \longrightarrow W_{\lambda_0} =$ 

$(W_{\lambda_0})_{red}$ = plane cubic with a node

$p \in \mathbb{P}^3$ general $\Rightarrow l_{\lambda_0} = \langle p, \underline{0} \rangle$

$\Rightarrow T_{\underline{0}} W_{\lambda_0} \ni p$

$\Rightarrow T_{\underline{0}} W_{\lambda_0} = \mathbb{P}^3$

$\Rightarrow W_{\lambda_0} \not\subset$ smooth cubic surface of $\mathbb{P}^3$!!

Theorem $\frac{1}{2}$

Every $X \in \mathcal{Y}_{14} \setminus \mathcal{Y}_8$ contains a smooth rational normal scroll T.

In particular X is rational

Pf $X \supset T_{\lambda_0}$ limit

of $\{T_\lambda\}_{\lambda \neq \lambda_0}$ T_λ rat. normal

scroll of

degree 4

$X \notin \mathcal{V}_8 \Longrightarrow (T_{\lambda_0})_{red}$

IRREDUCIBLE

T_λ OADP $\Longrightarrow T_{\lambda_0}$ NON

DEGENERATE

$$\begin{array}{ccc} T_\lambda & \longleftrightarrow & [C_\lambda] \subset \mathbb{G}(1,5) \\ \downarrow & & \downarrow \\ T_{\lambda_0} & & [C_{\lambda_0}] \end{array}$$

$\Longrightarrow (T_{\lambda_0})_{red}$ ruled by lines

(? IRREDUCIBLE)

(i) $T_{\lambda_0} = (T_{\lambda_0})_{red}$

$T_{\lambda_0} = T$ ✓

T_{λ_0} cone over a rational normal curve $\Longrightarrow X$ SINGULAR at the vertex of T_{λ_0}

(ii) $T_{X_0} \neq (T_{X_0})_{red}$

$\Rightarrow (T_{X_0})_{red}$ degenerated

& with an embedded point p_0

$\Rightarrow X$ singular at

p_0.

<u>CONTRADICTION</u>

Theorem (Bolognesi, – ; 2015/2016)

$X \in \mathcal{C}_{14} \setminus \mathcal{C}_8$ fixed

$\Longrightarrow Z = \{T \subset X\}$ is

a proper scheme

(indeed a projective surface!)

Pf Exactly as before.

NOW X fixed and

T varies inside X. $\square$

What is Z?

Theorem $X \in \mathcal{Y}_{14} \setminus \mathcal{Y}_8$

$\Longrightarrow$ Z smooth K3 surface of genus 8 & degree 14, i.e. a general section of $G(1,5) \subset \mathbb{P}^{14}$ with a $\mathbb{P}^8$

let us see this.

$$Y = \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$$

$$\{ \operatorname{rk} \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 & x_7 \end{bmatrix} = 1 \}$$

$$\phi : \mathbb{P}^7 \dashrightarrow \mathbb{G}(1,3) \subset \mathbb{P}^5$$

$$[A] \longmapsto \text{(Plücker coordinates of } A \in \mathrm{Mat}_{2\times 4}(\mathbb{K}))$$

$$\boxed{\mathrm{Bs}\,\phi = Y}$$

$$\phi^{-1}(\phi(p)) = \mathbb{P}^3_p \quad \forall\, p \in \mathbb{P}^7 \setminus Y$$

and $\mathbb{P}^3_p \cap Y = \mathbb{P}^1 \times \mathbb{P}^1_p$

$$T = \mathbb{P}^5 \cap Y \qquad \mathbb{P}^5 \subset \mathbb{P}^7 \text{ general}$$

$$\hat{\phi} = \phi|_{\mathbb{P}^5} : \mathbb{P}^5 \dashrightarrow \mathbb{G}(1,3)$$

Some of the fibers are linear spaces.

$\Rightarrow$ $\hat{\Phi}$ has general fiber a secant line to T

$X \supset T$ X smooth cubic 4-fold

$\hat{\Phi}_{|X} : X \dashrightarrow \mathbb{G}(1,3)$ birational

$\cup$

let $\hat{Z}$ base locus of $\hat{\Phi}_{|X}^{-1}$

$\hat{\Phi} : Bl_T X \longrightarrow \mathbb{G}(1,3)$ birational morphism

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$X \in \mathcal{C}_{14} \setminus \mathcal{C}_8 \Longrightarrow \hat{Z}$ smooth

$$\hat{Z} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{secant lines to } T \\ \text{contained in } X \end{array} \right\}$$

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liaison + geometry of surfaces (Fano or Alzati–Russo 2004)

$\Rightarrow \hat{Z} \simeq Bl_p Z$ with $Z \subset \mathbb{P}^8$

degree 14 & genus 8

smooth K3 surface

$Z \dashrightarrow \hat{Z}$ tangential projection from $p \in Z$

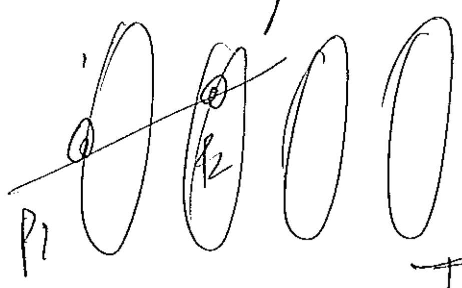
$$\begin{array}{ccc} Bl_p Z & \xrightarrow{\ \simeq\ } & \widehat{Z} \\ \cup & & \\ E & & \end{array}$$

E exc divisor

$\widehat{E} \subset \widehat{Z} \subset \mathbb{P}^5$

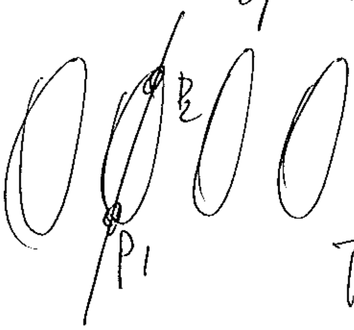
conic

general secant line



$T \simeq S(2,2)$

special secant line



$T \simeq S(2,2)$ (same for $S(1,3)$)

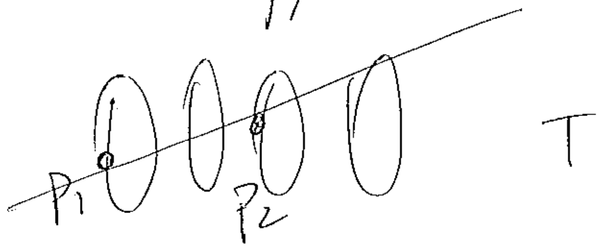
$$E \longleftrightarrow \left\{ \begin{array}{l} \text{special secant lines} \\ \text{to } \cancel{x} \text{ contained in } X \end{array} \right\}$$

$\left(\Sigma = \bigcup_{\lambda \in \mathbb{P}^1} \langle C_\lambda \rangle \quad C_\lambda \subset T \text{ conics} \quad \Sigma \cap X = T \cup R \quad R \text{ degree 5 scroll, union of special lines} \right)$

We now explain Fano's Deformation argument.

§ Fano's deformation argument

Let $L = \langle p_1, p_2 \rangle$, $p_i \in T$, be a secant line to T, which now is supposed to be general



Let $\pi_L : T \dashrightarrow Q_L \subset \mathbb{P}^3$ be the projection from L

$Q_L \simeq \mathbb{P}^1 \times \mathbb{P}^1$ quadric surface

Let $\{L_\lambda\}_{\lambda \in \Lambda_1 \simeq \mathbb{P}^1}$, $\{L_\mu\}_{\mu \in \Lambda_2 \simeq \mathbb{P}^1}$ be the rulings of Q_L.

Define $\{\mathbb{P}^3_\lambda\}_{\lambda \in \Lambda_1}$, $\{\mathbb{P}^3_\mu\}_{\mu \in \Lambda_2}$ with $\mathbb{P}^3_\lambda = \langle L, L_\lambda \rangle$, $\mathbb{P}^3_\mu = \langle L, L_\mu \rangle$.

Then

$F_\lambda = \mathbb{P}^3_\lambda \cap X \subset \mathbb{P}^3_\lambda$

are cubic surfaces

$G_\mu = \mathbb{P}^3_\mu \cap X \subset \mathbb{P}^3_\mu$

such that $L \subset F_\lambda \cap G_\mu$

We can suppose

$\mathbb{P}^3_\lambda \cap T = \widetilde{L}_\lambda$ line of the ruling of T

$\widetilde{L}_\lambda \subset F_\lambda$

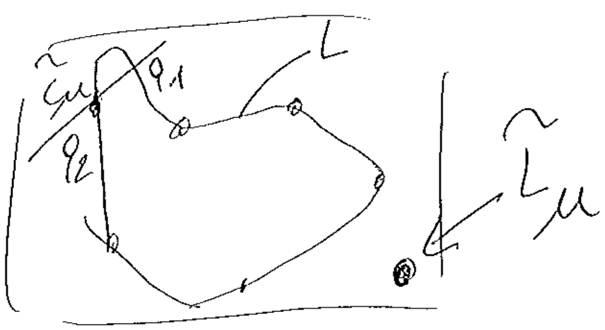
$\mathbb{P}^3_\mu \cap T = \widetilde{C}_\mu$ twisted cubic contained in T

$\widetilde{C}_\mu \subset G_\mu$.

Since $\pi_L(\widetilde{C}_\mu) = L_\mu \Rightarrow \widetilde{C}_\mu \cap L = \{q_1, q_2\}$

<u>Claim</u> $\exists!$ line $\widetilde{L}_\mu \subset G_\mu$:

$\widetilde{L}_\mu \cap \widetilde{C}_\mu = \phi = \widetilde{L}_\mu \cap L$



Define $\widehat{\widetilde{T}}_L = \bigcup_{\mu \in \Lambda_2} \widetilde{L}_\mu$ $\quad \left(T = \bigcup_{\lambda \in \Lambda_1} \widehat{L}_\lambda \right)$

$\widehat{\widetilde{T}}_L \subset X$ is a rational scroll

<u>Claim</u> $\widehat{\widetilde{T}}_L$ is a smooth rational normal scroll of degree 4.

Pf let $\overline{L} \subset X$ be a special secant line

let $L \longrightarrow \overline{L}$. In this degeneration Q_L degenerates to $Q_{\overline{L}} = \mathbb{T}_{\overline{L}}(T)$ which is a rank 3 quadric cone.

Thus Λ_1, Λ_2 degenerates to a unique ruling Λ and $\widehat{\widetilde{T}}_L \longrightarrow T$.

$$\begin{array}{ccc} \tilde{Z} = Bl_p Z & \longrightarrow & Z \\ \cup & & \\ E & & \end{array}$$

Thus the ~~spa~~ special secant lines to T provides rational normal scrolls equal to T and the contraction of E to p yields the moduli space.

This is rigorously proved by this beautiful result of Fano

Lemma (Fano) $T_1, T_2 \subset X$ algebraically equivalent smooth rational normal scrolls of degree 4 ($\Rightarrow T_1 \cdot T_2 = 10$).

$\Longrightarrow \exists!\ Q \subset \mathbb{P}^5$ quadric hyp. such that

(i) $T_1 \cup T_2 \subseteq Q$

(ii) $\mathrm{Vert}(Q) = L$, L secant line to T_1 and to T_2 ($\Rightarrow L \subset X$)

$\psi : Z^{(2)} \dashrightarrow F(X)$ Hilbert scheme of lines contained in X

$[(P_1, P_2)] \longrightarrow \mathrm{Vert}(Q)$

Z moduli space of quadrics contained in $X \in \mathcal{C}_{14} \setminus \mathcal{C}_8$
Z ~~is~~ is a K3 surface

Fano's deformation argument implies that ψ extends to an isomorphism

$\hat{\psi} : Bl_\Delta Z^{(2)} \longrightarrow F(X)$

blow-up symmetric product along the diagonal

Corollary (Beauville - Donagi isomorphism generalized)

$X \in \mathcal{C}_{14} \cap \mathcal{C}_8$

$F(X) = \left\{ \begin{array}{c} \text{Hilbert scheme of lines} \\ \text{contained in } X \end{array} \right\}$

$\Longrightarrow F(X) \simeq Z^{[2]} = Bl_{\Delta} Z^{(2)}$

symmetric product.

Proposition (Beauville) $X \in \mathcal{C}$

$X \supset S$ del Pezzo surface of degree 5 $\iff$ X is Pfaffian, i.e.

$$\exists\ A = \begin{array}{cc} & \begin{array}{cc} 3 & \end{array} \\ 3 & \left[\begin{array}{c|c} & \\ \hline & \end{array}\right] \end{array}$$

$= -A^t$ matrix of linear forms on $\mathbb{P}^5$:

$|A| = \mathrm{Pf}(A)^2$

$\mathrm{Pf}(A)$ cubic

$X = V(\mathrm{Pf}) \subset \mathbb{P}^5$

$$\mathrm{Pf} = \{ [X] \in \mathcal{C} : X \supset S \} \subset \mathcal{C}_{14}$$

Pfaffian locus

Beauville $\Longrightarrow$ (i) $\overline{\mathrm{Pf}} = \mathcal{C}_{14}$

(ii) $\dim \overline{\mathrm{Pf}} = 19$ (Beauville)

Theorem 2 (Bolognesi, –, Stagliano)
2015

a) Pf is dense but not open in $\mathcal{P}_{14}$

b) $\{X \in \mathcal{P}_{14} : X \supset T\}$ is dense but not open in $\mathcal{P}_{14}$