Categorical representability and rationality

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Semiorthogonal decompositions

- k is a field
- X a smooth and projective k-variety of dimension n
- $\mathbf{D}^{\mathrm{b}}(X)$ is a k-linear triangulated category

(nonsmooth cases = replace with categorical resolutions of singularities)

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Definition

A semiorthogonal decomposition of $\mathbf{D}^{\mathrm{b}}(X)$ is a set $\mathbf{A}_1, \ldots, \mathbf{A}_r$ s.t.:

- $\mathbf{A}_i \subset \mathbf{D}^{\mathrm{b}}(X)$ is admissible (i.e. full, thick and with adjoints)
- $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(A_j, A_i) = 0$ for all $j > i$, and A_* in $\mathbf{A}_*$
- $\mathbf{D}^{\mathrm{b}}(X)$ is the smallest triangulated category containing the $\mathbf{A}_i$'s.

We write: $\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{A}_1, \ldots, \mathbf{A}_r \rangle$.

Exceptional objects

Definition

For a simple k-algebra A, an object E in $\mathbf{D}^{\mathrm{b}}(X)$ is A-exceptional if

- $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(E, E) = A$
- $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(E, E[i]) = 0$ for $i \neq 0$

A sequence $E_1, \ldots, E_r$ is exceptional if E_i is A_i-exceptional for all i and

- $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(E_j, E_i[l]) = 0$ for $j > i$ and all l.

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- $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(E_j, E_i[l]) = 0$ for $j > i$ and all l.

Example. If X is Fano of index i, any line bundle is k-exceptional, and $\mathcal{O}, \ldots, \mathcal{O}(i-1)$ is an exceptional sequence.

Exceptional collections and Semiorthogonal decompositions

If E is A-exceptional in $\mathbf{D}^{\mathrm{b}}(X)$, then $\langle E \rangle \simeq \mathbf{D}^{\mathrm{b}}(A)$ is an admissible subcategory in $\mathbf{D}^{\mathrm{b}}(X)$.

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There is a semiorthogonal decomposition:

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{A}, E_1, \ldots, E_r \rangle$$

where $\mathbf{A}$ is the category of objects left-orthogonal to the E_i's.

Examples

Fanos of Picard rank one

Let X be a Fano variety of Picard rank one and index i. Then

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{A}_X, \mathcal{O}, \ldots, \mathcal{O}(i-1) \rangle.$$

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Mori fibrations

Let $p : X \to Y$ be a Mori fibration of relative index i. Then

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{A}_{X/Y}, p^*\mathbf{D}^{\mathrm{b}}(Y), \ldots, p^*\mathbf{D}^{\mathrm{b}}(Y)(i-1) \rangle.$$

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Blow up

If $X \to Y$ is the blow-up along a smooth Z of codimension d, then:

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{D}^{\mathrm{b}}(Y), \mathbf{D}^{\mathrm{b}}(Z)_1, \ldots, \mathbf{D}^{\mathrm{b}}(Z)_{d-1} \rangle.$$

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- [Beilinson, Orlov] if p is a projective bundle, then $\mathbf{A}_{X/Y} = 0$
- [Kuznetsov] if p is a quadric fibration, then $\mathbf{A}_{X/Y} = \mathbf{D}^{\mathrm{b}}(Y, \mathcal{C}_0)$, where $\mathcal{C}_0$ is the even Clifford algebra of p.

Categorical Representability

Definition

We say that X is *categorically representable* in dimension m (or equivalently in codimension $\dim(X) - m$) if there is a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{A}_1, \ldots, \mathbf{A}_r \rangle,$$

and smooth projective $Y_1, \ldots, Y_r$ of dimension $\leq m$ such that $\mathbf{A}_i$ is admissible in $\mathbf{D}^{\mathrm{b}}(Y_i)$.

We will use the following notations:

$$\begin{gathered}\mathrm{Rep}_{\mathrm{cat}}(X) := \{\text{min } m | X \text{ is c.rep. in dim } m\} \\ \mathrm{coRep}_{\mathrm{cat}}(X) := \dim(X) - \mathrm{Rep}\mathbf{D}^{\mathrm{b}}(X),\end{gathered}$$

and note that $\mathrm{Rep}_{\mathrm{cat}}(\mathbb{P}^n) = 0$ for any n, for any k.

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In all the Mori fiber space cases, the relevant information is in $\mathbf{A}_{X/Y}$.

The ring of triangulated categories

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$PT(k)$ is the $\mathbb{Z}$-module generated by triangulated categories $\mathbf{A}$ which are admissible in some $\mathbf{D}^{\mathrm{b}}(X)$, where we set

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad \text{if } \mathbf{C} = \langle \mathbf{A}, \mathbf{B} \rangle.$$

The product on $PT(k)$ is induced by $\mathbf{D}^{\mathrm{b}}(X) \bullet \mathbf{D}^{\mathrm{b}}(Y) = \mathbf{D}^{\mathrm{b}}(X \times Y)$, whence $PT(k)$ is a commutative ring with unit $\mathbf{e} = \mathbf{D}^{\mathrm{b}}(k)$.

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Categorical representability induces a ring filtration:

$$PT_d(k) := \langle \mathbf{D}^{\mathrm{b}}(X) \mid \mathrm{Rep}_{\mathrm{cat}}(X) \leq d \rangle^{+}$$

It is not know whether $\mathbf{D}^{\mathrm{b}}(X) \in PT_d(k)$ implies $\mathrm{Rep}_{\mathrm{cat}}(X) \leq d$.

Example We note that $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n) \in PT_0(k)$ for all n.

Motivic measures

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The Grothendieck ring of varieties $K_0(\mathrm{Var}(k))$ is the $\mathbb{Z}$-module generated by smooth and proper varieties where we set

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A motivic measure is a ring homomorphism $K_0(\mathrm{Var}(k)) \to R$.

We set $\mathbb{L} := [\mathbb{A}^1]$ the class of the affine line and note that $[\mathbb{P}^n] = \sum_{i=0}^{n} \mathbb{L}^i$.

Bondal-Larsen-Lunts and Larsen-Lunts measures

Larsen-Lunts' Measure

$$\mu : K_0(\mathrm{Var}(k)) \to SB(k)$$

sending $[X]$ to its stably birational equivalence class is a motivic measure whose kernel is the ideal $\langle \mathbb{L} \rangle$.

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Bondal-Larsen-Lunts' measure

$$\nu : K_0(\mathrm{Var}(k)) \to PT(k)$$

sending $[X] \to \mathbf{D}^{\mathrm{b}}(X)$ is a motivic measure.

Note that $\nu(\mathbb{L}) = \mathbf{e}$.

The noncommutative motivic rational defect

By Larsen-Lunts, and weak factorization, if X is rational, then:

$$[X] = [\mathbb{P}^n] + \mathbb{L} \sum M_X$$

in $K_0(\mathrm{Var}(k))$, where M_X are classes of varieties of dimension $\leq n-2$.

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in $K_0(\mathrm{Var}(k))$, where M_X are classes of varieties of dimension $\leq n-2$.
Applying ν, we obtain:

Proposition

If X rational then $\mathbf{D}^{\mathrm{b}}(X)$ is in $PT_{n-2}(k)$.

Proposition

Let $X \to Y$ be a Mori fibration with either Y rational or $\dim(Y) \leq n-2$. $\mathbf{D}^{\mathrm{b}}(X)$ is in $PT_{n-2}(k)$ if and only if $\mathbf{A}_{X/Y}$ is in $PT_{n-2}(k)$.

Categorical represented in low dimensions

Proposition

For any k, a k-linear triangulated category $\mathbf{A}$ is representable in dimension 0 if and only if there exists an étale k-algebra K and $\mathbf{A} \simeq \mathbf{D}^{\mathrm{b}}(K)$

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Proposition (Okawa)

For $k = \overline{k}$, a k-linear triangulated $\mathbf{A}$ is representable in dimension 1 but not 0 if and only if there is a smooth projective curve of genus $g > 0$ such that $\mathbf{A} \simeq \mathbf{D}^{\mathrm{b}}(C)$.

Note. A complete classification holds for any k as well.

Del Pezzo surfaces over arbitrary fields

k is any field, S a del Pezzo surface.

Theorem (Auel-B, 2015)

S is rational if and only if $\mathrm{Rep}_{\mathrm{cat}}(S) = 0$ if and only if $\mathbf{A}_S$ is representable in dimension 0.

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There exists a well-defined category $\mathbf{GK}_S$ which is a birational invariant. It is trivial if and only if S is rational.

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If $\deg(S) \geq 5$, then there are two vector bundles V_1, V_2 such that $A_i := \mathrm{End}(V_i)$ is semisimple and controls the birational geometry of S: from $c_2(V_i)$ and A_i we can construct $\mathbf{GK}_S$, and we can calculate the (arithmetical) index of S.

Del Pezzo surfaces over arbitrary fields

Sketch of the proof.

- If S is minimal, then $K_0(S) \simeq \mathbb{Z}^3$, then $\mathrm{Rep}_{\mathrm{cat}}(S) = 0$ implies $\mathbf{D}^{\mathrm{b}}(S) = \langle \mathbf{D}^{\mathrm{b}}(l_1), \mathbf{D}^{\mathrm{b}}(l_2), \mathbf{D}^{\mathrm{b}}(l_3) \rangle$ with l_i/k finite extension.

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- It follows that there is an exceptional collection of $\mathbf{D}^{\mathrm{b}}(\overline{S})$ in three blocks (chunks of mutually orthogonal exceptional bundles).

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- Karpov-Nogin et al. showed that there are only a finite number of such collections and that they are made of vector bundles. Consider the vector bundles V_1, V_2 and V_3 generating each block.

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- The descent of V_i's contradicts minimality if $\deg(S) \leq 4$, and in this case S is never rational. We set $\mathbf{GK}_S := \mathbf{A}_S = \langle \mathcal{O} \rangle^{\perp}$.

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- The descent of V_i's contradicts minimality if $\deg(S) \leq 4$, and in this case S is never rational. We set $\mathbf{GK}_S := \mathbf{A}_S = \langle \mathcal{O} \rangle^\perp$.
- If $\deg(S) \geq 5$, we have $V_3 = \mathcal{O}$, and V_1 and V_2 always descend, and X is rational if and only if $\mathrm{End}(V_i)$ is étale. We set $\mathbf{GK}_S$ to be the product of derived categories of those non-étale algebras.

Amitsur-type consequences

Proposition (Antieau)

If A, B are simple k-algebras with centers l_A and l_B, then $\mathbf{D}^{\mathrm{b}}(A) \simeq \mathbf{D}^{\mathrm{b}}(B)$ if and only if $l_A \simeq l_2$ and A and B have the same class in $\mathrm{Br}(l_A)$.

If S is minimal of degree $\deg(S) \geq 5$ the Brauer classes of A_1 and A_2 are a birational invariant.

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If S is minimal of degree $\deg(S) \geq 5$ the Brauer classes of A_1 and A_2 are a birational invariant.

Brauer–Severi case

If $S = BS(A)$ is a Brauer–Severi surface, and A has order 3. We have

$$A_1 = A \quad , \quad A_2 = A^2$$

so that our result is the Amitsur conjecture for surfaces.

Surfaces, phantoms and decomposition of the diagonal

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Recall: the latter means that there is a zero-cycle p on S such that $\Delta = p \times S + Z$ in $CH_2(S \times S)_{\mathbb{Z}}$ with Z supported on $S \times V$ and $\dim(V) < 2$. It is a necessary (but not sufficient) condition for stable rationality.

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Definition

An admissible subcategory $\mathbf{A}$ of some $\mathbf{D}^{\mathrm{b}}(X)$ is a phantom if $K_0(\mathbf{A}) = 0$ and $HH_*(\mathbf{A}) = 0$.

Proposition

Let S be a surface with a semiorthogonal decomposition

$$\langle \mathbf{A}, E_1, \ldots, E_r \rangle,$$

with E_i are k-exceptional and $\mathbf{A}$ a phantom.
Then S has an integral decomposition of the diagonal.

Surfaces, phantoms and decomposition of the diagonal

Sketch of the proof.

- If $\chi(\mathcal{O}_S) = 1$ and $K_0(S)_{\mathbb{Z}} \simeq \mathbb{Z}^n$ is generated by χ-semiorthogonal objects, then S a 0-cycle a of degree 1.

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- Under these assumptions $CH_0(S) = \mathbb{Z}[a]$, and $CH_1(S) = \mathrm{Pic}(S)$. Moreover, $K_0(S) \simeq CH_*(S)$ as a $\mathbb{Z}$-module.

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- Via GRR, transform the matrix of the bilinear form χ into the matrix of the intersection form on $\mathrm{Pic}(S)$. From the χ-semiorthogonality, we have a natural base D_i and unimodularity.

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- there are orthogonal projectors $a \times S$, $S \times a$ and $D_i \times D_i^\vee$, whose orthogonal is a idempotent cycle z on $S \times S$.

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- Base changing to any field, we obtain that the action of z is trivial on $CH_*(S_K)$, hence z is nilpotent, hence trivial.
- Conclusion: $\Delta = a \times S + S \times a + \sum D_i \times D_i^\vee$

Threefolds ans their Jacobians

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Recall: A_X is the maximal ppa subvariety not containing Jacobians of curves.

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Proposition (-,Bolognesi, 2011)

Suppose that X has $J(X)$ ppav with an incidence polarization. In this case if $\mathrm{Rep}_{\mathrm{cat}}(X) \leq 1$ we have $A_X = 0$.

Threefolds ans their Jacobians

$k = \mathbb{C}$. Let X be a Fano threefold, or a conic bundle over a a rational surface, or a del Pezzo fibration over $\mathbb{P}^1$.

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- having an incidence polarization is known in almost all cases.
- [-,Tabuada 2014] For a Mori fiber space $X \to Y$ as above, there is a well defined Jacobian $J(\mathbf{A}_{X/Y})$ as ppav, and $J(\mathbf{A}_{X/Y}) = J(X)$

Threefolds ans their Jacobians

Sketch of the proof.

- If there is a curve C such that $\mathbf{D}^{\mathrm{b}}(C) \to \mathbf{D}^{\mathrm{b}}(X)$ is admissible, then the functor is Fourier–Mukai, i.e. there is an object $\mathcal{E}$ in $\mathbf{D}^{\mathrm{b}}(X \times C)$ representing it.

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- If $\mathrm{Rep}_{\mathrm{cat}}(X) = 1$, then

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{D}^{\mathrm{b}}(C_1), \ldots, \mathbf{D}^{\mathrm{b}}(C_r), E_1, \ldots, E_s \rangle$$

so that $CH_*(X)_{alg=0,\mathbb{Q}} = \oplus_{i=1^r} \mathrm{Pic}^0(C_i)$ via GRR.

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- If $X \to Y$ is a rational conic bundle over a minimal surface [-,Bolognesi]

Then $\mathrm{Rep}_{\mathrm{cat}}(\mathbf{A}_{X/Y}) \leq 1$.

The Artin-Mumford double solid

Let $X \to \mathbb{P}^3$ be the Artin-Mumford double solid, and $W \to X$ the blow up of its 10 singular points. W is not rational but $J(W) = 0$.

Proposition

$\mathrm{Rep}_{\mathrm{cat}}(W) = 2.$

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We could also use a minimal resolution Z of X which is just a Moishezon manifold and use similar arguments.

Thank you!