Isogenies and transcendental Hodge structures of K3 surfaces

Samuel Boissière

Université de Poitiers
Mathématiques

Carry-Le-Rouet, May 2016

In collaboration with Alessandra Sarti and Davide Cesare Veniani

Isogenies

Definition

An isogeny $\gamma\colon X \dashrightarrow Y$ is a rational finite map of (prime) degree p between two complex projective K3 surfaces X and Y.

- Example:

$$\begin{tikzcd} X \arrow[loop left, "\sigma"] \arrow[d] \arrow[drr, dashed, "\gamma"] & & \\ X/\sigma & & Y \arrow[ll, "\text{resolution}"'] \end{tikzcd}$$

symplectic automorphism of order p

$\frac{24}{p+1}$ A_{p-1} singularities

Transcendental Hodge structure

X : complex projective K3 surface,
$\mathrm{H}^2(X,\mathbb{Z})$: lattice for the intersection product.
Néron–Severi lattice: $\mathrm{NS}(X) := \mathrm{H}^2(X,\mathbb{Z}) \cap \mathrm{H}^{1,1}(X)$,
Transcendental lattice: $\mathrm{T}_X := \mathrm{NS}(X)^{\perp}$.

Proposition

The Hodge structure on $\mathrm{H}^2(X,\mathbb{C})$ induces a Hodge decomposition

$$\mathrm{T}_{X,\mathbb{C}} = \mathrm{T}_X^{2,0} \oplus \mathrm{T}_X^{1,1} \oplus \mathrm{T}_X^{0,2}$$

which is an irreducible Hodge structure of weight two.

- Aim of the talk:

Relation between T_X and T_Y when X and Y are isogenous?

Why this is interesting: the Hodge conjecture

M : projective manifold,
Group of Hodge classes : $\mathrm{Hdg}^k(M) := \mathrm{H}^{2k}(M,\mathbb{Q}) \cap \mathrm{H}^{k,k}(M)$.

Hodge conjecture

Every Hodge class is algebraic, *i.e.* is a linear combination with rational coefficients of algebraic subvarieties.

- In this talk:
  - $M := Y \times X$ is 4-dimensional
  - The Hodge conjecture is open for the group of Hodge classes

$$\mathrm{Hdg}^2(M) = \mathrm{H}^4(M,\mathbb{Q}) \cap \mathrm{H}^{2,2}(M).$$

Interesting Hodge classes in $\mathrm{Hdg}^2(Y \times X)$

$$
\begin{array}{rcl}
\xi & \in & \mathrm{Hdg}^2(Y \times X) \cap (\mathrm{H}^2(Y,\mathbb{Q}) \otimes \mathrm{H}^2(X,\mathbb{Q})) \\
 & \Updownarrow & \\
\xi & \in & \mathrm{Hom}_{\mathrm{Hdg}}\left(\mathrm{H}^2(Y,\mathbb{Q}), \mathrm{H}^2(X,\mathbb{Q})\right) \\
\| & & \quad\quad\quad\wr\| \\
\xi_{\mathrm{NS}} & \in & \mathrm{Hom}_{\mathrm{Hdg}}\left(\mathrm{NS}(Y)_{\mathbb{Q}}, \mathrm{NS}(X)_{\mathbb{Q}}\right) \\
+ & & \quad\quad\quad\oplus \\
\xi_{\mathrm{T}} & \in & \mathrm{Hom}_{\mathrm{Hdg}}\left(\mathrm{T}_{Y,\mathbb{Q}}, \mathrm{T}_{X,\mathbb{Q}}\right)
\end{array}
$$

- As a consequence:

$$\xi \text{ is algebraic } \Leftrightarrow \xi_{\mathrm{T}} \text{ is algebraic}$$

The Hodge conjecture for products of K3 surfaces

Conjecture

Let X, Y be two complex projective K3 surfaces. Every morphism of rational Hodge structure $\mathrm{T}_{Y,\mathbb{Q}} \to \mathrm{T}_{X,\mathbb{Q}}$ is algebraic.

Known results

1. (Mukai 1987) True if $\mathrm{T}_{Y,\mathbb{Q}} \to \mathrm{T}_{X,\mathbb{Q}}$ is an isometry and $\rho(Y) \geq 11$.
2. (Nikulin 1987) True if $\mathrm{T}_{Y,\mathbb{Q}} \to \mathrm{T}_{X,\mathbb{Q}}$ is an isometry and $\rho(Y) \geq 5$.
3. (Buskin 2015) True if $\mathrm{T}_{Y,\mathbb{Q}} \to \mathrm{T}_{X,\mathbb{Q}}$ is an isometry.

- Aim of the talk:

What if $\mathrm{T}_{Y,\mathbb{Q}} \to \mathrm{T}_{X,\mathbb{Q}}$ is not an isometry?

Isogenous K3 surfaces

- Resolution of the indeterminacies of the isogeny:

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\beta"'] \arrow[dr, "\tilde{\gamma}"] & \\ X \arrow[rr, dashed, "\gamma"'] & & Y \end{tikzcd}$$

β birational, $\tilde{\gamma}$ generically finite

rational map of degree p

- Example:

$$\begin{tikzcd} X \arrow[loop left, "\sigma"] \arrow[d] \arrow[dr, dashed, "\gamma"] & \widetilde{X} \arrow[l, "\beta"'] \arrow[d, "\tilde{\gamma}"] \\ X/\sigma & Y \arrow[l, "\text{resolution}"'] \end{tikzcd}$$

symplectic automorphism of order $p = 2$

β blowup, $\tilde{\gamma}$ ramified double covering

Correspondence induced by an isogeny

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\beta"'] \arrow[dr, "\tilde{\gamma}"] & \\ X \arrow[rr, dashed, "\gamma"'] & & Y \end{tikzcd}$$

$$\Gamma := \mathrm{Im}\left((\beta, \tilde{\gamma}) \colon \widetilde{X} \to X \times Y\right) \subset X \times Y$$

The algebraic correspondence by Γ is a morphism

$$\gamma^* \colon \mathrm{H}^2(Y, \mathbb{Z}) \to \mathrm{H}^2(X, \mathbb{Z})$$

Proposition (Inose)

$\gamma^* \colon \mathrm{T}_Y \to \mathrm{T}_X$ is a dilation with scale factor p.

Interpretation

- An isogeny $\gamma\colon X \dashrightarrow Y$ of primer order p induces a morphism of rational Hodge structures $\gamma^*\colon \mathrm{T}_{Y,\mathbb{Q}} \to \mathrm{T}_{X,\mathbb{Q}}$ which is not an isometry, but *still* algebraic.
- More subtle question: even if γ^* is not an isometry, might the rational quadratic spaces $\mathrm{T}_{Y,\mathbb{Q}}$ and $\mathrm{T}_{X,\mathbb{Q}}$ still be isometric?
- *Different formulation*:

  Can non-isometric rational transcendental Hodge structures of K3 surfaces be related by an algebraic correspondence?

Main result

Theorem (-, Sarti, Veniani)

Let $\gamma\colon X \dashrightarrow Y$ be an isogeny of prime degree p between two complex projective K3 surfaces. Denote $r := \operatorname{rk} \mathrm{T}_Y = \operatorname{rk} \mathrm{T}_X$.

1. If r is odd, $\mathrm{T}_{X,\mathbb{Q}}$ and $\mathrm{T}_{Y,\mathbb{Q}}$ are never isometric.
2. If r is even, $\mathrm{T}_{X,\mathbb{Q}}$ and $\mathrm{T}_{Y,\mathbb{Q}}$ are isometric iff:
   - $p = 2$: for any prime number $q \equiv 3$ or $5 \mod 8$, the q-adic valuation $\nu_q(\det \mathrm{T}_X)$ is even.
   - $p > 2$:
     - for any odd prime number $q \neq p$ such that p is not a square in $\mathbb{F}_q$, $\nu_q(\det \mathrm{T}_X)$ is even;
     - $\operatorname{res}_p(\det \mathrm{T}_X) = (-1)^{\frac{r(r-1)}{2} + \nu_p(\det \mathrm{T}_X)} \in \mathbb{F}_p^*/(\mathbb{F}_p^*)^2$.

- Recall: for any $p^\alpha \frac{s}{t} \in \mathbb{Q}$, $\operatorname{res}_p\left(p^\alpha \frac{s}{t}\right) = \frac{s}{t} \in \mathbb{F}_p$.
- This generalizes a previous result of van Geemen–Sarti (2007) for isogenies induced by symplectic involutions.

Examples: isogenies induced by symplectic automorphisms

$$\begin{tikzcd} X \arrow[loop left, "\sigma"] \arrow[d] \arrow[dr, dashed, "\gamma"] & \\ X/\sigma & Y \arrow[l, "\text{resolution}"'] \end{tikzcd}$$

symplectic automorphism of order p

$\frac{24}{p+1}$ A_{p-1} singularities

- *Generically* $\operatorname{rk} \mathrm{NS}(Y) = \frac{24}{p+1}(p-1) + 1$

$$r = \operatorname{rk} \mathrm{T}_X = \operatorname{rk} \mathrm{T}_Y = 21 - \frac{24}{p+1}(p-1) \geq 2$$

- $p \in \{2, 3, 5, 7\}$
- *Generically* r is odd, so $\mathrm{T}_{X,\mathbb{Q}}$ and $\mathrm{T}_{Y,\mathbb{Q}}$ are not isometric.

Example: Shioda–Inose structure

$$\begin{tikzcd} X \arrow[loop left, "\sigma"] \arrow[d] \arrow[dr, dashed, "\gamma"] & \\ X/\sigma & Y \arrow[l, "\text{resolution}"'] \end{tikzcd}$$

symplectic involution such that

- Y is a Kummer surface
- $\gamma^* \colon \mathrm{T}_Y(2) \to \mathrm{T}_X$ isometry over $\mathbb{Z}$

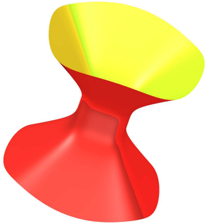
Theorem (Morrison)

If $\rho(X) \in \{19, 20\}$ then X admits a Shioda–Inose structure.

- If $\rho(X) = 19$, $\mathrm{T}_{X,\mathbb{Q}}$ and $\mathrm{T}_{Y,\mathbb{Q}}$ are never isometric.
- If $\rho(X) = 20$, both cases can occur.

Example: the Fermat quartic

$$X : x^4 + y^4 + z^4 + t^4 = 0$$



$$\mathrm{T}_X = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \sim_{\mathbb{Q}} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

- $p = 2$: symplectic involution $\sigma(x, y, z, t) = (-x, -y, z, t)$. One has $\det \mathrm{T}_X = 2^2$, $\nu_q(\det \mathrm{T}_X) = 0 \quad \forall q \equiv 3, 5 \mod 8$ so:

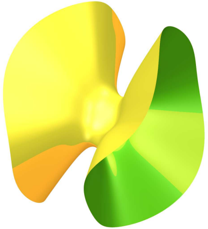
  $\mathrm{T}_{Y,\mathbb{Q}}$ *is isometric to* $\mathrm{T}_{X,\mathbb{Q}}$.
- $p = 3$: automorphism $\sigma(x, y, z, t) = (y, z, x, t)$. One has $\nu_q(\det \mathrm{T}_X) = 0$ if $q \neq 3$ and $2^2 \neq (-1) \in \mathbb{F}_3^* / (\mathbb{F}_3^*)^2$ so:

  $\mathrm{T}_{Y,\mathbb{Q}}$ *is not isometric to* $\mathrm{T}_{X,\mathbb{Q}}$.

Example: the Schur quartic

$$X : x^4 - xy^3 = z^4 - zt^3$$

$$\mathrm{T}_X = \begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix} \sim_{\mathbb{Q}} \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$$



- $p = 2$: symplectic involution $\sigma(x, y, z, t) = (-x, y, -z, t)$. One has $\det \mathrm{T}_X = 3 \cdot 2^2$, $\nu_3(\det \mathrm{T}_X) = 1$ so:

  $\mathrm{T}_{Y,\mathbb{Q}}$ *is not isometric to* $\mathrm{T}_{X,\mathbb{Q}}$.

- $p = 3$: automorphism $\sigma(x, y, z, t) = (x, \zeta_3 y, z, \bar{\zeta}_3 t)$. One has $\mathrm{T}_{X,\mathbb{Q}}(3) \sim_{\mathbb{Q}} \langle 2, 6 \rangle \sim_{\mathbb{Q}} \mathrm{T}_{X,\mathbb{Q}}$ so:

  $\mathrm{T}_{Y,\mathbb{Q}}$ *is isometric to* $\mathrm{T}_{X,\mathbb{Q}}$.

Elements of proof

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\beta"'] \arrow[dr, "\tilde{\gamma}"] & \\ X \arrow[rr, dashed, "\gamma"'] & & Y \end{tikzcd}$$

$$\gamma^* \colon \mathrm{T}_Y \to \mathrm{T}_X$$
dilation with scale factor p

1. $\mathrm{T}_Y(p) \hookrightarrow \mathrm{T}_X$ is a sublattice of the same rank
2. $[\mathrm{T}_X, \mathrm{T}_Y(p)]^2 = p^r \left|\frac{\det \mathrm{T}_Y}{\det \mathrm{T}_X}\right|$
3. r odd:
   If $\mathrm{T}_{X,\mathbb{Q}} \cong \mathrm{T}_{Y,\mathbb{Q}}$ then $\left|\frac{\det \mathrm{T}_Y}{\det \mathrm{T}_X}\right| \in (\mathbb{Q}^*)^2$, impossible since r is odd.
4. r even: one has $\mathrm{T}_{Y,\mathbb{Q}}(p) \cong \mathrm{T}_{X,\mathbb{Q}}$ so

$$\mathrm{T}_{X,\mathbb{Q}} \cong \mathrm{T}_{Y,\mathbb{Q}} \Longleftrightarrow \mathrm{T}_{X,\mathbb{Q}}(p) \cong \mathrm{T}_{X,\mathbb{Q}}$$

- A complete answer is provided by Witt theory.

Witt theory

Witt group $W(\mathbb{K})$: Witt-equivalence classes of regular quadratic $\mathbb{K}$-forms.

$$\partial_q(r) := \begin{cases} 0 & \text{if } \nu_q(r) \text{ is even} \\ \mathrm{res}_q(r) & \text{if } \nu_q(r) \text{ is odd} \end{cases}$$

$$\bar{\partial}_q \colon W(\mathbb{Q}) \to W(\mathbb{F}_q), \langle a_1, \ldots, a_n \rangle_W \mapsto \langle \partial_q(a_1), \ldots, \partial_q(a_n) \rangle_W$$

Theorem

The morphisms $\bar{\partial}_q$ induce a group isomorphism

$$W(\mathbb{Q}) \cong \bigoplus_{q \text{ prime}, q=\infty} W(\mathbb{F}_q)$$

Two numerical consequences:

Proposition

Two regular quadratic $\mathbb{Q}$-forms φ and ψ are Witt-equivalent iff they have the same signature over $\mathbb{R}$ and for any prime number q one has

$$\bar{\partial}_q([\varphi]_W) = \bar{\partial}_q([\psi]_W) \in W(\mathbb{F}_q).$$

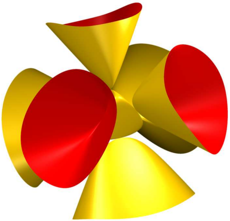
Discriminant $\Delta(\varphi) := (-1)^{\frac{n(n-1)}{2}} \det(\varphi) \in \mathbb{K}/(\mathbb{K}^*)^2, \quad n = \dim \varphi.$

Proposition

Two regular quadratic $\mathbb{F}_q$-forms are Witt-equivalent iff their dimensions have the same parity and their discriminants are equal.

- These propositions provide a numerical criterium to decide whether $\mathrm{T}_{X,\mathbb{Q}}(p) \cong \mathrm{T}_{X,\mathbb{Q}}$ or not.

Example: isogeny not induced by an automorphism



- A : abelian surface
- $\mathrm{Km}(A)$: Kummer surface of A
- $G \subset A[p]$: subgroup of order p

of the group of p-torsion points of A.

$$\begin{tikzcd} A \arrow[r, "p:1"] \arrow[d, dashed] & B = A/G \arrow[d, dashed] \\ \mathrm{Km}(A) \arrow[r, dashed, "\gamma"] & \mathrm{Km}(B) \end{tikzcd}$$

For $p > 7$, the map γ is an isogeny between two K3 surfaces, which cannot be induced by a symplectic automorphism.

Example: isometry not coming from an isogeny

- X: projective K3 surface
- Mukai vector $v = (r, \ell, s) \in \widetilde{\mathrm{H}}^*(X, \mathbb{Z})$ (Mukai lattice), ℓ ample
- $M_v(X)$: moduli space of ℓ-stable vector bundles E on X with Mukai vector $v(E) = \mathrm{ch}(E)\sqrt{\mathrm{td}(X)} = v$.

Theorem (Mukai)

If $v^2 = 0$, $r, s > 0$, $\gcd(r, s) = 1$, then $M_v(X)$ is a projective K3 surface.

- Universal family

$$\begin{tikzcd} & \mathcal{E} \arrow[d] & \\ & X \times M_v(X) \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_M"] & \\ X & & M_v(X) \end{tikzcd}$$

$M := M_v(X)$

$$\begin{tikzcd} & \mathcal{E} \arrow[d] & \\ & X \times M \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_M"] & \\ X & & M \end{tikzcd}$$

integral algebraic class:
$Z := \pi_X^* \sqrt{\mathrm{td}(X)}\,\mathrm{ch}(\mathcal{E})\pi_M^*\sqrt{\mathrm{td}(M)}$

isometry:
$[Z]_* \colon v^\perp/\mathbb{Z}v \to \mathrm{H}^2(M,\mathbb{Z})$

restricts to an exact sequence

$$0 \to \mathrm{T}_X \xrightarrow{\varphi} \mathrm{T}_M \to \mathrm{Coker}(\varphi) \to 0$$

inducing a rational isometry $\mathrm{T}_{X,\mathbb{Q}} \cong \mathrm{T}_{M,\mathbb{Q}}$.

Proof of Mukai–Nikulin theorem

Mukai–Nikulin–Buskin theorem

Let X, Y be two complex projective K3 surfaces. If $\rho(Y) \geq 5$ then every isometry of rational Hodge structure $\varphi\colon \mathrm{T}_{Y,\mathbb{Q}} \to \mathrm{T}_{X,\mathbb{Q}}$ is algebraic.

Why is it important to assume that φ is an isometry?

- The easy case: true if $\varphi\colon \mathrm{T}_Y \to \mathrm{T}_X$ is an isometry over $\mathbb{Z}$ and $\rho(Y) > 11$.
- New geometric idea (moduli spaces of sheaves on K3): true if $\varphi\colon \mathrm{T}_Y \to \mathrm{T}_X$ is an isometry over $\mathbb{Z}$, no restriction on $\rho(Y)$.
- Induction argument: true if $\varphi\colon \mathrm{T}_Y \to \mathrm{T}_X$ is defined over $\mathbb{Z}$ and has a finite cokernel. Induction on the number of generators of Coker φ, no restriction on $\rho(Y)$.
- General case: the isometry $\varphi\colon \mathrm{T}_{Y,\mathbb{Q}} \to \mathrm{T}_{X,\mathbb{Q}}$ is not defined over $\mathbb{Z}$ and $\rho(Y) \geq 11$ (Mukai), or $\rho(Y) \geq 5$ (Nikulin).

The easy case

$$\begin{tikzcd} \mathrm{T}_Y \arrow[r, hook] \arrow[d, "\varphi"'] & \mathrm{H}^2(Y,\mathbb{Z}) \arrow[d, dashed, "\Phi"] \\ \mathrm{T}_X \arrow[r, hook] & \mathrm{H}^2(X,\mathbb{Z}) \end{tikzcd}$$

$\exists\Phi$ extending φ
if $\rho(Y) > 11$ (Nikulin)

- $\tau_d\colon \mathrm{H}^2(Y,\mathbb{Z}) \to \mathrm{H}^2(Y,\mathbb{Z})$: reflection by a (-2)-class d.
- Torelli theorem: $\Phi \circ \tau_{d_1} \circ \cdots \circ \tau_{d_k} = f^*$, where $f\colon X \to Y$ is an isomorphism.
- $\varphi = \Phi_{|\mathrm{T}_Y} = f^*_{|\mathrm{T}_Y}$ since $\mathrm{T}_Y \perp d_i$.
- so φ is the correspondence by the graph of f, hence is algebraic.

General case

$\varphi\colon \mathrm{T}_{Y,\mathbb{Q}} \to \mathrm{T}_{X,\mathbb{Q}}$ is an isometry, not defined over $\mathbb{Z}$.

- Consider the sublattice $T := \mathrm{T}_Y \cap \varphi^{-1}(\mathrm{T}_X)$.
- $\varphi_T\colon T \hookrightarrow \mathrm{T}_X$ is an isometry over $\mathbb{Q}$, already defined over $\mathbb{Z}$.
- If there exists an embedding of T in the K3-lattice, whose orthogonal complement is hyperbolic, then by the *Surjectivity of the Period Map* there exists a K3 surface S such that $\mathrm{T}_S \cong T$.
- $\varphi_1\colon \mathrm{T}_S \to \mathrm{T}_X$ is algebraic (induction step)
- $\varphi_2\colon \mathrm{T}_S \to \mathrm{T}_Y$ is algebraic (induction step)
- so $\varphi = \varphi_2 \circ \varphi_1^{-1}$ is algebraic
- the condition is fulfilled if $\rho(T) < 11$, or better if $\rho(Y) \geq 5$ (Nikulin).

Speculation

Hodge conjecture

Let X, Y be two complex projective K3 surfaces. Every morphism of rational Hodge structures $\mathrm{T}_{Y,\mathbb{Q}} \to \mathrm{T}_{X,\mathbb{Q}}$ is algebraic.

Mukai–Nikulin–Buskin theorem

Let X, Y be two complex projective K3 surfaces. Every isometry of rational Hodge structures $\mathrm{T}_{Y,\mathbb{Q}} \to \mathrm{T}_{X,\mathbb{Q}}$ is algebraic.

Speculation

Let X, Y be two complex projective K3 surfaces. Every dilation of rational Hodge structures $\mathrm{T}_{Y,\mathbb{Q}} \to \mathrm{T}_{X,\mathbb{Q}}$ is algebraic.