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HOMOLOGICAL PROJECTIVE DUALITY FOR DETERMINANTAL VARIETIES (with M.Bernardara and D.Faenzi)

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Semi-orthogonal decompositions

Let X be a smooth projective algebraic variety.

Definition

A semiorthogonal decomposition of $\mathbf{D^b(X)}$ *is an ordered sequence of full admissible triangulated subcategories* $\mathbf{A_1}, \dots, \mathbf{A_n}$ *of* $\mathbf{D^b(X)}$ *such that :*

- $\mathrm{Hom}_{\mathbf{D^b(X)}}(\mathbf{A_i}, \mathbf{A_j}) = \mathbf{0}$ *for all* $i > j$ *and*
- *for all objects* A_i *of* $\mathbf{A_i}$ *and* A_j *of* $\mathbf{A_j}$ *and for every object* T *of* $\mathbf{D^b(X)}$*, there is a chain of morphisms* $0 = T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 = T$ *such that the cone of* $T_k \to T_{k-1}$ *is an object of* $\mathbf{A_k}$ *for all* $1 \leq k \leq n$.

Such a decomposition will be written

$$\mathbf{D^b(X)} = \langle \mathbf{A_1}, \dots, \mathbf{A_n} \rangle.$$

SOD and exceptional objects

Definition

An object E of $\mathbf{D^b(X)}$ is exceptional if $\mathrm{Hom}_{\mathbf{D^b(X)}}(E, E) = \mathbb{C}$ and $\mathrm{Hom}_{\mathbf{D^b(X)}}(E, E[i]) = 0$ for all $i \neq 0$. An ordered sequence $(E_1, \ldots, E_l)$ of exceptional objects is an exceptional collection if $\mathrm{Hom}_{\mathbf{D^b(X)}}(E_j, E_k[i]) = 0$ for all $j > k$ and for all $i \in \mathbb{Z}$.

SOD is a strong property, there exists varieties that have no SOD at all.

- Calabi-Yau varieties
- postive genus curves

The importance of being SOD

- SODs seem to encode the birational geometry of algebraic varieties (Ex : cubic 3fold and Fano X_{14})
- rationality questions (Ex : conic bundles, cubic 4folds)
- Fano visitor problem

Birational geometry

$X_3 \subset \mathbb{P}^4$ the smooth cubic threefold.
$X_{14} = Gr(2,6) \cap \mathbb{P}^9 \subset \mathbb{P}^{14}$ the smooth Fano 3fold of degree 14.
Any $\widetilde{X_{14}}$ is birational to a given $\widetilde{X_3}$ [Fa,I-M,...]

$$\mathbf{D^b}(\widetilde{\mathbf{X_3}}) = \langle \mathbf{A_3}, \mathcal{O}(-\mathbf{1}), \mathcal{O} \rangle;$$

$$\mathbf{D^b}(\widetilde{\mathbf{X_{14}}}) = \langle \mathbf{A_{14}}, \mathbf{E}, \mathcal{O} \rangle;$$

with $\mathbf{A}_3 \overset{equiv}{\cong} \mathbf{A}_{14}$. Many other instances of this phenomenon for Fano 3folds. [Kuz]

Rationality questions

Theorem (B-B)

A standard conic bundle X over a rational minimal surface is rational iff $\mathbf{D^b(X)} = \langle \mathbf{D^b(C_1)}, \ldots, \mathbf{D^b(C_m)}, \mathbf{E_1}, \ldots, \mathbf{E_r} \rangle$.

If you prefer, iff X is categorically representable in dimension 1.

Definition (B-B)

A triangulated category $\mathbf{T}$ *is representable in dimension j if it admits a semiorthogonal decomposition*

$$\mathbf{T} = \langle \mathbf{A_1}, \ldots, \mathbf{A_l} \rangle,$$

and for all $i = 1, \ldots, l$ there exists a smooth projective connected variety Y_i with $\dim Y_i \leq j$, *such that* $\mathbf{A_i}$ *is equivalent to an admissible subcategory of* $\mathbf{D^b(Y_i)}$.

Rationality questions

If Y is a smooth cubic 4fold, then

$$\mathbf{D^b(Y)} = \langle \mathbf{B}, \mathcal{O}(-\mathbf{2}), \mathcal{O}(-\mathbf{1}), \mathcal{O} \rangle.$$

Conjecture (Kuznetsov)

Y is rational iff $\mathbf{B} \cong \mathbf{D^b(S)}$ for S a smooth projective K3 surface.

Question (B-B)

Is a rational projective variety always categorically representable in codimension 2 ?

Fano visitor problem

Any smooth Fano variety admits a semi-orthogonal decomposition.

Question (Bondal)

Let X be a smooth projective variety. Does there exists a smooth Fano variety Y and a fully faithful functor $\mathbf{D^b(X)} \to \mathbf{D^b(Y)}$ *?*

But in general : it is very hard to construct a SOD for $\mathbf{D^b(X)}$. So far there exists only one effective way to find SOD, that is *Homological Projective Duality*.

What is HPD ?

HPD is a generalization of classical projective duality that involves SODs. These are of a particular kind, that is sensible to linear sections

Definition

A Lefschetz decomposition of $\mathbf{D^b(X)}$ *with respect to* $\mathcal{O}_X(H)$ *is a semiorthogonal decomposition*

$$\mathbf{D^b(X)} = \langle \mathbf{A_0}, \mathbf{A_1(H)}, \ldots, \mathbf{A_{i-1}((i-1)H)} \rangle, \tag{1}$$

with $0 \subset \mathbf{A_{i-1}} \subset \ldots \subset \mathbf{A_0}$*. Such a decomposition is said to be rectangular* if $\mathbf{A_0} = \ldots = \mathbf{A_{i-1}}$.

$$\mathbf{D^b(X \cap H)} = \langle \mathbf{A_1(H)}, \ldots, \mathbf{A_{i-1}((i-1)H)} \rangle$$

What is HPD ?

Let $W := H^0(X, \mathcal{O}_X(H))$, and $f : X \to \mathbb{P}W$ s.t. $f^*\mathcal{O}_{\mathbb{P}W}(1) \cong \mathcal{O}_X(H)$. Let $\mathcal{X} \subset X \times \mathbb{P}W^\vee$ be the universal hyperplane section of X

$$\mathcal{X} := \{(x, H) \in X \times \mathbb{P}W^\vee | x \in H\}.$$

Definition

Let $f : X \to \mathbb{P}W$ be a smooth projective scheme with a base-point-free line bundle $\mathcal{O}_X(H)$ and a Lefschetz decomposition as above. A scheme Y with a map $g : Y \to \mathbb{P}W^\vee$ is called homologically projectively dual (or the HP-dual) to $f : X \to \mathbb{P}W$ with respect to the Lefschetz decomposition (1)*, if there exists a fully faithful functor $\Phi : \mathbf{D^b(Y)} \to \mathbf{D^b}(\mathcal{X})$ giving the semiorthogonal decomposition*

$$\mathbf{D^b}(\mathcal{X}) = \langle \mathbf{\Phi(D^b(Y))}, \mathbf{A_1(1)} \boxtimes \mathbf{D^b}(\mathbb{P}\mathbf{W}^\vee), \ldots, \mathbf{A_{i-1}(i{-}1)} \boxtimes \mathbf{D^b}(\mathbb{P}\mathbf{W}^\vee) \rangle.$$

HPD's consequences

Let $N = \dim(W)$, $c \leq N$ be an integer, $L \subset W$ a c-dimensional linear space. $\mathbb{P}_L := \mathbb{P}(W/L) \subset \mathbb{P}W$. Dually $\mathbb{P}^L = \mathbb{P}_L^{\perp} \subset \mathbb{P}W^{\vee}$.

$$X_L := X \times_{\mathbb{P}W} \mathbb{P}_L, \qquad Y_L := Y \times_{\mathbb{P}W^{\vee}} \mathbb{P}^L.$$

Theorem (K)

If Y is HP-dual to X, then :

(i) *Y is smooth projective and admits a dual Lefschetz decomposition*

$$\mathbf{D^b(Y)} = \langle \mathbf{B_{j-1}(1-j)}, \ldots, \mathbf{B_1(-1)}, \mathbf{B_0} \rangle, \quad \mathbf{B_{j-1}} \subset \ldots \subset \mathbf{B_1} \subset \mathbf{B_0}$$

with respect to $\mathcal{O}_Y(H) = g^*\mathcal{O}_{\mathbb{P}W^{\vee}}(1)$.

HPD's consequences

(ii) if L is admissible, i.e. if

$$\dim X_L = \dim X - c, \quad \text{and} \quad \dim Y_L = \dim Y + c - N,$$

then there exist a triangulated category $\mathbf{C_L}$ and semiorthogonal decompositions :

$$\mathbf{D^b(X_L)} = \langle \mathbf{C_L}, \mathbf{A_c(1)}, \ldots, \mathbf{A_{i-1}(i-c)} \rangle,$$
$$\mathbf{D^b(Y_L)} = \langle \mathbf{B_{j-1}(N-c-j)}, \ldots, \mathbf{B_{N-c}(-1)}, \mathbf{C_L} \rangle.$$

Think for example if $c > i$...
Like classical projective duality, HPD involves singular varieties.
More precisely :

Theorem (K)

The critical locus of the map $g : Y \to \mathbb{P}W^\vee$ is the classical projective dual $X^\vee$ of X.

Categorical resolutions of singularities

The presence of singular varieties requires another piece of our categorical tool kit, that is categorical resolutions of singularities.

Definition

A categorical resolution of singularities $(X, \mathcal{A})$ *of a possibly singular proper scheme* X *is a torsion free* $\mathcal{O}_X$*-algebra* $\mathcal{A}$ *of finite rank such that* $\mathrm{Coh}(X, \mathcal{A})$ *has finite homological dimension (i.e., is smooth in the noncommutative sense).*

If X (or Y) is singular, then we replace $\mathbf{D^b(X)}$ (or $\mathbf{D^b(Y)}$) by a categorical resolution of singularities $\mathbf{D^b}(\mathbf{X}, \mathcal{A})$ (or $\mathbf{D^b}(\mathbf{Y}, \mathcal{B})$). The theorems hold in this more general framework, where we consider $\mathbf{D^b}(\mathbf{X_L}, \mathcal{A}_\mathbf{L})$ (resp. $\mathbf{D^b}(\mathbf{Y_L}, \mathcal{B}_\mathbf{L})$) for $\mathcal{A}_L$ (resp. $\mathcal{B}_L$) the restriction of $\mathcal{A}$ to X_L (resp. of $\mathcal{B}$ to Y_L)

Categorical resolutions : a clarification

There is a plethora of different definitions of categorical (or noncommutative) resolution of singularities. In this work, the sheaf that will provide a non-commutative resolution of singular determinantal varieties, will resolve them in the sense fo Van den Bergh, since the sheaf will be locally the endomorphism algebra of a reflexive sheaf. This also implies that it is a resolution in the sense of Kuznetsov (cf. Roland's talk).

Determinantal varieties and Springer resolution

Let U, V be vector spaces, $\dim U = m$, $\dim V = n$, $n \geq m$. Set $W = U \otimes V$. Let $r \in \mathbb{Z}$ s.t. $1 \leq r \leq m-1$. We define $\mathcal{Z}^r_{m,n}$ to be the variety of matrices $M : V \to U^\vee$ in $\mathbb{P}W$ cut by the minors of size $r+1$ of the matrix of indeterminates :

$$\psi = \begin{pmatrix} x_{1,1} & \dots & x_{m,1} \\ \vdots & \ddots & \vdots \\ x_{m,n} & \dots & x_{m,n} \end{pmatrix}$$

i.e. the rank r locus.

Determinantal varieties and Springer resolution

Let $\mathbb{G}(U, r)$ be the Grassmannian of r-dimensional quotients of U. Let $\mathcal{U}$ be the tautological sub-bundle and $\mathcal{Q}$ the quotient bundle over $\mathbb{G}(U, r)$, resp. of rank $m - r$ and r. The Euler sequence reads :

$$0 \to \mathcal{U} \to U \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{Q} \to 0. \tag{2}$$

We will use the following notation :

$$\mathcal{X}^r_{m,n} = \mathbb{P}(V \otimes \mathcal{Q}).$$

The manifold $\mathcal{X}^r_{m,n}$ has dimension $r(n + m - r) - 1$. It is the resolution of singularities of the variety $\mathcal{Z}^r_{m,n}$ of $m \times n$ matrices of rank at most r. It is commonly known as the *Springer resolution*.

Determinantal varieties and Springer resolution

Let $p : \mathcal{X}^r_{m,n} \to \mathbb{G}(U, r)$ be the natural projection. $H^0(\mathbb{G}, \mathcal{Q})$ is identified with U. Let $\mathcal{O}(H)$ be the relatively ample tautological line bundle on $\mathcal{X}^r_{m,n}$. We get :

$$H^0(\mathbb{G}, V \otimes \mathcal{Q}) \simeq H^0(\mathcal{X}^r_{m,n}, \mathcal{O}(H)) \simeq W = U \otimes V.$$

The map f associated with $\mathcal{O}(H)$ maps $\mathcal{X}^r_{m,n}$ to $\mathbb{P}W$, and $\mathcal{O}(H) \simeq f^*(\mathcal{O}_{\mathbb{P}W}(1))$. That is :

$$\begin{array}{ccc}
\mathcal{X}^r_{m,n} & \xrightarrow{f} & \mathbb{P}W = \mathbb{P}(U \otimes V) \\
{\scriptstyle p}\downarrow & & \\
\mathbb{G} & &
\end{array}$$

Determinantal varieties and Springer resolution

One can show that f is birational, the image is exactly $\mathcal{Z}^r_{m,n}$ and, being an isomorphism over the matrices of rank exactly r, f gives a desingularization of $\mathcal{Z}^r_{m,n}$.

More concretely, let $\lambda \in \mathbb{G}$ and π_λ be the linear projection from $U^\vee$ to $U^\vee/\mathcal{Q}^\vee_\lambda$. Then, the variety $\mathcal{X}^r_{m,n}$ can be seen as :

$$\mathcal{X}^r_{m,n} = \{(\lambda, M) \in \mathbb{G} \times \mathcal{Z}^r_{m,n} \mid \pi_\lambda \circ M = 0\}.$$

Then, p and f are just the projections from $\mathcal{X}^r_{m,n}$ onto the two factors.

Determinantal varieties and Springer resolution

Dually, consider the projective bundle :

$$\mathcal{Y}^r_{m,n} = \mathbb{P}(V^\vee \otimes \mathcal{U}^\vee).$$

Let $q : \mathcal{Y}^r_{m,n} \to \mathbb{G}$ be the projection. Abusing notation, we denote by H the tautological ample line bundle on $\mathcal{Y}^r_{m,n}$. Since

$H^0(\mathbb{G}, \mathcal{U}^\vee) \simeq U^\vee$, the linear system $|\mathcal{O}_{\mathcal{Y}^r_{m,n}}(H)|$ sends $\mathcal{Y}^r_{m,n}$ to $\mathbb{P}W^\vee \simeq \mathbb{P}(V^\vee \otimes U^\vee)$ via a map g. The map g is a desingularization of the variety $\mathcal{Z}^{m-r}_{m,n}$ of matrices $V^\vee \to U$ in $\mathbb{P}W^\vee$ of corank at least r.

The non-commutative resolution

Consider $p : \mathcal{X}^r_{m,n} \to \mathbb{G}$ as rank$-(rn-1)$ projective bundle. Hence [Or] we have :

$$\mathbf{D^b}(\mathcal{X}^{\mathbf{r}}_{\mathbf{m,n}}) = \langle \mathbf{p}^*\mathbf{D^b}(\mathbb{G}), \ldots, \mathbf{p}^*\mathbf{D^b}(\mathbb{G})((\mathbf{rn-1})\mathbf{H})\rangle. \qquad (3)$$

$\mathbb{G}$ has a full strong exceptional collection [Ka] consisting of vector bundles, thus we obtain an exceptional collection on $\mathcal{X}^r_{m,n}$ consisting of vector bundles, and hence a tilting bundle E as the direct sum of these bundles. Let us set $M := Rf_*E$, and let $\mathcal{R} := \mathcal{E}nd(E)$ and $\mathcal{R}' := \mathcal{E}nd(M)$ (where $\mathcal{E}nd$ denotes the sheaf of endomorphisms).

Proposition

The endomorphism algebra $\mathcal{E}nd(M)$ is a coherent $\mathcal{O}_{\mathcal{Z}^r_{m,n}}$-algebra Morita-equivalent to $\mathcal{R}$. In particular, $\mathbf{D^b}(\mathcal{Z}^{\mathbf{r}}_{\mathbf{m,n}}, \mathcal{R}') \simeq \mathbf{D^b}(\mathcal{X}^{\mathbf{r}}_{\mathbf{m,n}})$ is a categorical resolution of singularities, which is crepant if $m = n$.

HPD as projective bundles

Thanks to the preceding description, we can prove HPD directly from [Kuz]'s HPD for the projective bundles $\mathcal{X}^r_{m,n}$ and $\mathcal{Y}^r_{m,n}$. We consider the rectangular Lefschetz decomposition (3) for $\mathcal{X}^r_{m,n}$ with respect to $\mathcal{O}(H)$.

Theorem

The morphism $g : \mathcal{Y}^r_{m,n} \to \mathbb{P}W^\vee$ is the homological projective dual of $f : \mathcal{X}^r_{m,n} \to \mathbb{P}W$, relatively over $\mathbb{G}$, with respect to the rectangular Lefschetz decomposition (3) *induced by $\mathcal{O}(H)$, generated by $nr\binom{m}{r}$ exceptional bundles.*

HPD - a sketch of a proof.

Recall that $\mathcal{X}^r_{m,n} = \mathbb{P}(V \otimes \mathcal{Q})$ and $\mathcal{Y}^r_{m,n} = \mathbb{P}(V^\vee \otimes \mathcal{U}^\vee)$ are projective bundles over $\mathbb{G}$. Set $\mathbf{A} = \mathbf{p}^*(\mathbf{D^b}(\mathbb{G}))$. The decomposition (3) of the projective bundle $\mathcal{X}^r_{m,n} \to \mathbb{G}$ then reads :

$$\mathbf{D^b}(\mathcal{X}^{\mathbf{r}}_{\mathbf{m,n}}) = \langle \mathbf{A}, \mathbf{A}(\mathbf{H}), \dots, \mathbf{A}((\mathbf{rn}-\mathbf{1})\mathbf{H}) \rangle.$$

This is a rectangular Lefschetz decomposition with respect to $\mathcal{O}(H)$, generated by nr copies of [Ka]'s exceptional collection on $\mathbb{G}$, hence by $nr\binom{m}{r}$ exceptional bundles.
$V \otimes \mathcal{Q}$ and $V^\vee \otimes \mathcal{U}^\vee$ are generated by their global sections, so we apply apply [Kuz]. The evaluation map of $V \otimes \mathcal{Q}$ gives :

$$0 \to V \otimes \mathcal{U} \to W \to V \otimes \mathcal{Q} \to 0.$$

Hence $V^\vee \otimes \mathcal{U}^\vee$ is the orthogonal of $V \otimes \mathcal{Q}$ in [Kuz]'s sense. Therefore [Corollary 8.3, Kuz] applies and gives the result. Note that $\mathbf{D^b}(\mathcal{Y}^{\mathbf{r}}_{\mathbf{m,n}})$ is generated by $n(m-r)\binom{m}{r}$ exceptional vector bundles.

HPD - non commutative sense

This can be rephrased in terms of categorical resolutions. Hence one can state HPD as a duality between categorical resolutions of determinantal varieties given by matrices of fixed rank and corank.

Theorem

There is a $\mathcal{O}_{\mathcal{Z}^r_{m,n}}$-algebra $\mathcal{R}'$ such that $(\mathcal{Z}^r_{m,n}, \mathcal{R}')$ is a categorical resolution of singularities of $\mathcal{Z}^r_{m,n}$. Moreover, $\mathbf{D^b}(\mathcal{Z}^{\mathbf{r}}_{\mathbf{m,n}}, \mathcal{R}') \simeq \mathbf{D^b}(\mathcal{X}^{\mathbf{r}}_{\mathbf{m,n}})$ so that $(\mathcal{Z}^r_{m,n}, \mathcal{R}')$ is HP-dual to $(\mathcal{Z}^{m-r}_{m,n}, \mathcal{R}')$.

Consequences : semi-orthogonal decompositions of linear sections

Recall the notation : Let now c be an integer s.t. $1 \leq c \leq mn$, and suppose we a have c-dimensional vector subspace L of W :

$$L \subset U \otimes V = W.$$

We have the linear subspace $\mathbb{P}_L \subset \mathbb{P}W$ of codimension c, defined by $\mathbb{P}_L = \mathbb{P}(W/L)$ and dually $\mathbb{P}^L = \mathbb{P}L^\perp$ of dimension $c-1$ in $\mathbb{P}W^\vee$, whose defining equations are the elements of $L^\perp \subset W^\vee$. We define the varieties :

$$X_L^r = \mathcal{X}_{m,n}^r \times_{\mathbb{P}W} \mathbb{P}_L, \qquad Y_L^r = \mathcal{Y}_{m,n}^r \times_{\mathbb{P}W^\vee} \mathbb{P}^L.$$

We also write :

$$Z_L^r = \mathcal{Z}_{m,n}^r \cap \mathbb{P}_L, \qquad Z_r^L = \mathcal{Z}_{m,n}^{m-r} \cap \mathbb{P}^L.$$

We will always assume that $L \subset W$ is an *admissible subspace*, which amounts to ask that X_L and Y_L have expected dimension.

Consequences : semi-orthogonal decompositions of linear sections

Suppose that $L \subset W$ is admissible of dimension c.

(i) If $c > nr$, there is a fully faithful functor

$$\mathbf{D^b}(\mathbf{Z_L}, \mathcal{R}'_{\mathbb{P}_\mathbf{L}}) \simeq \mathbf{D^b}(\mathbf{X_L}) \longrightarrow \mathbf{D^b}(\mathbf{Y_L}) \simeq \mathbf{D^b}(\mathbf{Z_r^L}, \mathcal{R}'_{\mathbb{P}^\mathbf{L}})$$

whose complement is $c - nr$ copies of $\mathbf{D^b}(\mathbb{G})$.

(ii) If $nr = c$, there is an equivalence

$$\mathbf{D^b}(\mathbf{Z_L}, \mathcal{R}'_{\mathbb{P}_\mathbf{L}}) \simeq \mathbf{D^b}(\mathbf{X_L}) \simeq \mathbf{D^b}(\mathbf{Y_L}) \simeq \mathbf{D^b}(\mathbf{Z_r^L}, \mathcal{R}'_{\mathbb{P}^\mathbf{L}})$$

(iii) If $c < nr$, there is a fully faithful functor

$$\mathbf{D^b}(\mathbf{Z_r^L}, \mathcal{R}'_{\mathbb{P}^\mathbf{L}}) \simeq \mathbf{D^b}(\mathbf{Y_L}) \longrightarrow \mathbf{D^b}(\mathbf{X_L}) \simeq \mathbf{D^b}(\mathbf{Z_L}, \mathcal{R}'_{\mathbb{P}_\mathbf{L}})$$

whose complement is $nr - c$ copies of $\mathbf{D^b}(\mathbb{G})$.

Consequences : semi-orthogonal decompositions of linear sections

Call ϕ_K the canonical map of X_L. Write also ϕ_{-K} for the anticanonical map. P is the pull-back to $\mathbb{P}(V \otimes \mathcal{Q})$ of $c_1(\mathcal{Q})$.

Lemma

The canonical bundle of the linear section X_L is :

$$\omega_{X_L} \simeq \mathcal{O}_{X_L}((c - nr)H + (n - m)P).$$

i) *The variety X_L is Calabi-Yau if and only if $m = n$ and $c = nr$.*

ii) *If $c > nr$, or if $c = nr$ and $n > m$, ϕ_K is a birational morphism onto its image.*

iii) *If $c < nr$ and $m = n$, ϕ_{-K} is a birational morphism onto its image. If moreover $X_L^{r-1} = \emptyset$, ϕ_{-K} is an embedding and X_L is Fano.*

HPD - Table of the functors

An analogous Lemma holds for $\mathcal{Y}^r_{m,n}$ and its linear section. The following table resumes the results.

	$c < nr$	$c = nr$	$c > nr$
Functor	$\mathbf{D^b(Y_L)} \to \mathbf{D^b(X_L)}$	equivalence	$\mathbf{D^b(X_L)} \to \mathbf{D^b(Y_L)}$
$Y_L \to Z^L$	nef ω_{Y_L}	nef ω_{Y_L} *if* n$\neq m$	
	F. visitor if $n = m$	CY if $n = m$	Fano if $n = m$
$X_L \to Z_L$		nef ω_{X_L} *if* n$\neq m$	nef ω_{X_L}
	Fano if $n = m$	CY if $n = m$	F. visitor if $n = m$

Table – Behaviour of HPD functors according on c and nr.

Birational and equivalent linear sections

The condition $c = nr$ guarantees that HPD gives an equivalence of categories. Hence we obtain derived equivalences of Calabi-Yau manifolds for any $n = m$. Moreover a little argument shows :

Proposition

If $c = nr$ then X_L is birational to Y_L.

If X_L is Calabi-Yau, then $m = n$ and $c = nr$. Thus, in dimension 3, the derived equivalences would follow also from the work of [Bri].

Birational and equivalent linear sections

Question

Aren't X_L and Y_L isomorphic ?

There are infinitely many examples of birational, derived equivalent, non isomorphic X_L and Y_L, all of dimension at least 5. In all of them the canonical system is birational onto a hypersurface of general type contained in $\mathbb{G}$. First concrete example : $(r, m, n) = (3, 5, 7)$, $c = 21$. The determinantal model of X_L (respectively, of Y_L) is the fivefold of degree 490 (respectively, 1176) cut in $\mathbb{P}^{13}$ (respectively, in $\mathbb{P}^{20}$) by the 4×4 minors (respectively, the 3×3 minors) of a sufficiently general 5×7 matrix of linear forms.

Segre-determinantal duality

If $r = 1$ then $\mathcal{X}^1_{m,n}$ is just the Segre product $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ and $\mathcal{Y}^1_{m,n}$ is the Springer desingularization of the space of degenerate matrices.
In this case we have the following table of functors, where also rationality of varieties is highlighted.

	$c < m$	$m \leq c < n$	$c = n$	$n < c$
Functor	$\mathbf{D^b(Y_L)} \to \mathbf{D^b(X_L)}$		equivalence	$\mathbf{D^b(X_L)} \to \mathbf{D^b(Y_L)}$
Y_L	F. visitor		CY if $n = m$	Rational Fano if $n = m$
X_L	Rational Fano	Rational	CY if $n = m$	F. visitor if $n = m$

Table – The Segre-determinantal duality.

Representability into Fano varieties, Fano visitors

Question

X smooth, projective. Is there any smooth Fano Y with a full and faithful functor $\mathbf{D^b(X)} \to \mathbf{D^b(Y)}$?

A weakly Fano-visitor is a variety (or more generally a triangulated category) whose derived category is FF-embedded in the categorical resolution of singularities of a Fano variety.

Proposition

Suppose that $n = m$. If $c < rn$, then Y_L^r and $(Z_r^L, \mathcal{R}'_{\mathbb{P}^L})$ are weakly Fano visitor. If $c > nr$, then X_L^r and $(Z_L^r, \mathcal{R}'_{\mathbb{P}_L})$ are weakly Fano visitor.

Corollary

Let $Z \subset \mathbb{P}^k$ be a determinantal variety associated to a generic $m \times n$ matrix. If $k < m - 1$ then the categorical resolution of singularities of Z is Fano visitor.

Curves visiting Fano varieties

This corollary gives a positive answer to the Fano visitor problem for many classes of curves.

Plane curves.

Let $C \subset \mathbb{P}^2$ be a plane curve of degree $d \geq 4$. C can be written as the determinant of a $d \times d$ matrix of linear forms. We set $m = n = d$, $k = 2$. Hence any plane curve of degree at least four is a Fano-visitor, up to resolution of singularities. Since the blow-up of $\mathbb{P}^3$ along a plane cubic is Fano [BL], any plane curve of positive genus is a Fano-visitor.

More curves of general type

Determinantal varieties with $n \neq m$ provide examples of (non plane) curves of general type that are Fano-visitor. Setting $\dim(Y_L^1) = \dim(Z^L) = 1$, we have $c = n - m + 3$. Then Y_L^1 is an elliptic curve - a plane cubic - if $m = n = c = 3$. If $m = 2$ then the curve is rational for all n since $c = n + 1$, and if $m > 3$ it is a curve of general type in $\mathbb{P}^{c-1}$, which is Fano visitor if $c < m$.

An update

[Kiem-Kim-Lee-Lee] The derived category of a smooth complete intersection variety is equivalent to a full subcategory of the derived category of a smooth projective Fano variety.

Categorical representability

Definition

A triangulated category $\mathbf{T}$ *is representable in dimension* j *if it admits a semiorthogonal decomposition*

$$\mathbf{T} = \langle \mathbf{A_1}, \dots, \mathbf{A_l} \rangle,$$

and for all $i = 1, \dots, l$ *there exists a smooth projective connected variety* Y_i *with* $\dim Y_i \leq j$*, such that* $\mathbf{A_i}$ *is equivalent to an admissible subcategory of* $\mathbf{D^b}(\mathbf{Y_i})$.

Question

Is a rational projective variety always categorically representable in codimension at least 2 ?

Categorical representability

Corollary

The categorical resolution of a rational determinantal variety ($r = 1$) is categorically representable in codimension at least 2. A rational linear section of the Segre variety $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \subset \mathbb{P}^{nm-1}$ is categorically representable in codimension at least 2.

Categorical resolution of the residual category of a determinantal Fano hypersurface

The Segre-determinantal HPD involves categorical resolutions for determinantal varieties, crepant if $n = m$. For Fano varieties, such resolution may give a crepant categorical resolution for nontrivial components of a semiorthogonal decomposition. We assume here $r = 1$ and $m = n$.

Proposition

Let F be a projective Gorenstein variety with rational singularities. Suppose that $\mathrm{Pic}(F) = \mathbb{Z}$, $\mathcal{O}_F(1)$ is its (ample) generator and $K_F = \mathcal{O}_F(-i)$, with $i > 0$. Then there is a semiorthogonal decomposition

$$\mathbf{D^b}(\mathbf{F}) = \langle \mathcal{O}_{\mathbf{F}}(-\mathbf{i}+\mathbf{1}), \ldots, \mathcal{O}_{\mathbf{F}}, \mathbf{T}_{\mathbf{F}} \rangle.$$

This holds in particular if $F \subset \mathbb{P}^k$ is an hypersurface of degree $d < k$ with rational singularities (in which case, $i = k - d + 1$).

Categorical resolution of the residual category of a determinantal Fano hypersurface

Homological Projective Duality allows us to describe a resolution of singularities of $\mathbf{T_F}$ in the case where F is determinantal.

Proposition

Let Z be a Fano determinantal hypersurface of $\mathbb{P}^k$, $p : Y \to Z$ its resolution, and X the dual section of the Segre variety. There is a strongly crepant categorical resolution $\widetilde{\mathbf{T}}_Z$ of $\mathbf{T_Z}$, admitting a semiorthogonal decomposition by $\mathbf{D^b(X)}$ and $(d-1)(k-d+1)$ exceptional objects.

In fact by mutating we obtain :

$$\mathbf{D^b}(\mathbf{Y}) = \langle \mathbf{p}^*\mathcal{O}_\mathbf{Z}(-\mathbf{k}+\mathbf{d}), \ldots, \mathbf{p}^*\mathcal{O}_\mathbf{Z}, \mathbf{E_1}, \ldots, \mathbf{E_{(k-d+1)(d-1)}}, \mathbf{D^b}(\mathbf{X})\rangle.$$

Determinantal cubic 3folds and 4folds

For determinantal cubic 3folds and 4folds, the dual "Segre" linear section X is empty. So, the numeric values give explicitly :

- If Z is a determinantal cubic threefold, then the category $\mathbf{T}_Z$ admits a crepant categorical resolution of singularities generated by 4 exceptional objects.
- If Z is a determinantal cubic fourfold, then the category $\mathbf{T_Z}$ admits a crepant categorical resolution of singularities generated by 6 exceptional objects.

In the case of cubic threefolds and fourfolds with only one node, categorical resolution of singularities of $\mathbf{T_Z}$ are described [Kuz]. We expect that these geometric descriptions carry over to the more degenerate case of determinantal cubics - which are all singular.

Determinantal cubic 3folds and 4folds - Expectations

In the 3-dim. case, the 4 exc. obj. should correspond to a disjoint union of two rational curves, arising as the geometrical resolution of the discriminant locus of a projection $Z \to \mathbb{P}^3$ from one of the 6 singular points. This discriminant locus is composed by two twisted cubics intersecting in 5 points, a degeneration of the $(3, 2)$ complete intersection curve from the one-node case.

In the 4-dim. case, the 6 exc. obj. should correspond to a disjoint union of two Veronese-embedded planes (projected to $\mathbb{P}^4$), arising as the geometrical resolution of the discriminant locus of a projection $Z \to \mathbb{P}^4$ from one singular point. This discriminant locus is composed by two cubic scrolls intersecting along a quintic elliptic curve, a degeneration of the degree 6 K3 from the one-node case [Kuz].

Thank you !

THANK YOU !