Spaces of matrices of constant rank via cones of morphisms and truncated graded modules

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Intro & set up

$\mathrm{M}_{a,b}(\mathbf{k})$: matrices of size $a \times b$ over a field $\mathbf{k}$

A vector subspace $V \subseteq \mathrm{M}_{a,b}(\mathbf{k})$ (of dimension $n+1$) is a
space of matrices of constant rank
if all its nonzero elements have fixed rank r.

Main questions:

1. Determine $\ell(a, b, r)$: max dim V, and give relations among the possible values of the parameters a, b, r, n.
2. Classification for fixed values of a, b, r, n.
3. **Construction of examples.**

Spaces of matrices $\rightsquigarrow$ vector bundles

We look at the $n+1$-dim'l space V as an $a \times b$ matrix whose entries are linear forms (a *linear matrix*), and interpret the cokernel as a vector space varying smoothly over $\mathbb{P}^n$, i.e. a vector bundle!

We get an exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & \mathcal{O}_{\mathbb{P}^n}(-m-1)^b \arrow[rr, "V"] \arrow[dr, two heads] & & \mathcal{O}_{\mathbb{P}^n}(-m)^a \arrow[r] & E \arrow[r] & 0 \\ & & & \mathcal{E} \arrow[ur, hook] & & & \end{tikzcd}$$

with K, $\mathcal{E}$, and E v.b. on $\mathbb{P}^n$ of rank $b-r$, r, and $a-r$ resp.

Remark. The idea of using v.b. to study and construct linear matrices of constant rank dates back to [Sylvester '86].

Some results

- [Westwick '87] $b - r + 1 \leq \ell(a, b, r) \leq a + b - 2r + 1$ for $2 \leq r \leq a \leq b$ (**not sharp**)

- [Ellia-Menegatti '15] effective value of $\ell(a, r)$ for $a \leq 10$

- [Ilic-Landsberg '99] $\ell_{sym}(r + 2, r) = 3$

- [Manivel-Mezzetti '05] classification of spaces of skew-symm matrices of size 6 and constant rank 4, $\ell_{skew}(6, 4) = 3$

- [Fania-Mezzetti '11] classification of spaces of skew-symm matrices of size 8 and constant rank 6, $\ell_{skew}(8, 6) = 3$

- [B.-Mezzetti '15] classification of m-effective pairs (c_1, c_2) + **explicit** construction for all pairs

(m-effective: there exists a 3-dim'l space of skew-symm matrices of size $2c_1 + 2$, constant rank $2c_1$, whose cokernel E has $c_i(E) = c_i$)

Looking for explicit examples

Possible approaches:

- ⋆ "ad hoc" construction, e.g. [Westwick '90], or
- ⋆ projection from bigger size matrices.

Why projecting? The smaller $n - r$ is, the easier it becomes to find indecomposable rank r bundles on $\mathbb{P}^n$,

so a good strategy consists in first building a bigger matrix, and then projecting it to a smaller one of the same constant rank.

Cutting down columns (rows) $\leftrightarrow$ taking a quotient of K (resp. E)

Example. All m-effective bundles appearing in [B.-Mezzetti '15] are quotients of bundles of the form:

$$(\oplus_{i\geq 1}\mathcal{O}_{\mathbb{P}^2}(i)^{a_i}) \oplus Q^b \qquad \text{with } i \geq 1,\ a_i, b \geq 0 \text{ for all } i.$$

Small corank

The projection technique was also used in [Fania-Mezzetti '11], but it does not work for "small corank":

Proposition. Let V be a linear space of $a \times b$ matrices of constant rank r, and $\dim = n+1$. V induces by projection a space V' of $\alpha \times \beta$ matrices of the same constant rank r and dimension $n+1$ for any $\alpha \geq r+n$ and $\beta \geq r+n$.

Corollary. $n+1$-dim'l spaces of matrices of constant rank **cannot be constructed via projection** as soon as $n > \min\{\alpha - r, \beta - r\}$.

We say that an $n+1$-dim'l space of $a \times b$ matrices of constant rank r has *small corank* if $n > \min\{a - r, b - r\}$.

Vector bundles $\rightsquigarrow$ spaces of matrices?

[B.-Faenzi-Mezzetti '13]: method to prove existence of families of 4-dim'l spaces of skew-symmetric matrices of constant corank 2 (so small corank!)

Restart from the exact sequence:

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-m-1)^b \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-m)^a \longrightarrow E \longrightarrow 0$$

Vector bundles $\rightsquigarrow$ spaces of matrices?

[B.-Faenzi-Mezzetti '13]: method to prove existence of families of 4-dim'l spaces of skew-symmetric matrices of constant corank 2 (so small corank!)

Restart from the exact sequence in the skew-simm. case:

$$0 \to E^*(-2m-1) \to \mathcal{O}_{\mathbb{P}^3}(-m-1)^{r+2} \to \mathcal{O}_{\mathbb{P}^3}(-m)^{r+2} \to E \to 0$$

We get a distinguished element

$$\eta \in \mathrm{Ext}^2(E, E^*(-2m-1)) \simeq \mathrm{Hom}_{D^b(\mathbb{P}^3)}(E, E^*(-2m-1)[2]).$$

For all $t \in \mathbb{Z}$, the cup product with η induces maps:

$$\mu_t : \mathrm{H}^0(E(t)) \longrightarrow \mathrm{H}^2(E^*(-2m-1+t)).$$

Our distinguished elt. gives non-degeneracy conditions on the μ_t's.

Vector bundles $\rightsquigarrow$ spaces of matrices?

Idea: take $Cone(\eta)$, the cone of the morphism η, and decompose it with respect to the standard exceptional collection $\langle \mathcal{O}_{\mathbb{P}^3}(-m-2), \mathcal{O}_{\mathbb{P}^3}(-m-1), \mathcal{O}_{\mathbb{P}^3}(-m), \mathcal{O}_{\mathbb{P}^3}(-m+1)\rangle$.

Theorem I. (B.-Faenzi-Mezzetti)

E: rank 2 vector bundle on $\mathbb{P}^3$ "with the right invariants".

The **existence of a distinguished elt.** $\eta \in \mathrm{Ext}^2(E, E^*(-2m-1))$ inducing the non-degeneracy conditions on the maps μ_t's

is equivalent to

the **existence of a linear matrix** V, having size $r+2$, constant rank r, and a twist of E as its cokernel.

Such matrix appears as differential in the decomposition of $Cone(\eta)$, and moreover it is **necessarily skew-symmetrizable**.

Instanton bundles & spaces of matrices

The non-degeneracy conditions in **Theorem I** are hard to check!

It can be done in the following cases:

Theorem II. (B.-Faenzi-Mezzetti)

- Any 2-instanton on $\mathbb{P}^3$ induces a skew-symmetric matrix of linear forms in 4 variables having size 10 and constant rank 8.
- General 4-instantons on $\mathbb{P}^3$ induce a skew-symmetric matrix of linear forms in 4 variables having size 14 and constant rank 12.

Problem: these results are **non-constructive**!

Looking for a constructive algorithm

Remember: we want a construction vector bundle $\rightsquigarrow$ linear matrix.

So let E: v.b. on $\mathbb{P}^n = \mathit{Proj}(R)$, $\quad R = \mathbf{k}[x_0, \dots, x_n]$.

First idea: look at its minimal free resolution. Even better, resolve its module of sections: $\mathbf{E} = \mathrm{H}^0_*(E) = \bigoplus_{t \in \mathbb{Z}} \mathrm{H}^0(E(t))$.

It is a graded R-module with graded Betti numbers $\beta_{i,j}$ and minimal free resolution:

$$\cdots \longrightarrow \bigoplus_{j_1} R(-j_1)^{\beta_{1,j_1}} \longrightarrow \bigoplus_{j_0} R(-j_0)^{\beta_{0,j_0}} \longrightarrow \mathbf{E}$$

This is of course **too naïve**. The resolution is not *linear* in general!

Linear presentation, m-linearity

A graded R-module $\mathbf{E}$ has *m-linear resolution* over R if its minimal graded free resolution, for suitable integers $\beta_{i,m+i}$, is of type:

$$\cdots \longrightarrow R(-m-2)^{\beta_{2,m+2}} \longrightarrow R(-m-1)^{\beta_{1,m+1}} \longrightarrow R(-m)^{\beta_{0,m}} \longrightarrow \mathbf{E}$$

That is, $\mathbf{E}$ has a m-linear resolution if:

1. $\mathbf{E}_r = 0$ for $r < m$,
2. $\mathbf{E}$ is generated by $\mathbf{E}_m$, and
3. $\mathbf{E}$ has a resolution where all the maps are represented by matrices of **linear forms**.

$\mathbf{E}$ is *m-linearly presented up to order k*, or just *linearly presented* when $k = 1$, if only the first k maps are matrices of linear forms.

Truncated modules and regularity

We can get around the non-linearity of the resolution of $\mathbf{E}$ by

truncating the graded module "in the right spot",

namely its regularity $\mathrm{reg}(\mathbf{E})$.
(Can be computed as $\max\{j - i \mid \beta_{i,j} \neq 0\}$.)

Indeed, if $\mathbf{E} = \bigoplus_{t \in \mathbb{Z}} \mathbf{E}_t$, then $\mathbf{E}_{\geq m} = \bigoplus_{t \geq m} \mathbf{E}_t$ and $\mathbf{E}_{\geq \mathrm{reg}(\mathbf{E})}$ always has m-linear resolution.

So we got linearity, but we still need something a bit more sophisticated! To begin with, we **lost control over the size** of the matrix (in general it will be too big!)

(Vague) idea: "cut off a piece" of the linear matrix from the linear resolution of $\mathbf{E}_{\geq \mathrm{reg}(\mathbf{E})}$, **without modifying the rank**.

Cutting off pieces of a linear resolution

Construction from [B.-Faenzi-Lella '16]: let $\mathbf{E}$ and $\mathbf{G}$ be f.g. graded R-modules with minimal graded free resolutions:

$$\cdots \longrightarrow E^1 \xrightarrow{e_1} E^0 \xrightarrow{e_0} \mathbf{E}$$

$$\cdots \longrightarrow G^1 \xrightarrow[g_1]{} G^0 \xrightarrow[g_0]{} \mathbf{G}$$

Cutting off pieces of a linear resolution

Construction from [B.-Faenzi-Lella '16]: let $\mathbf{E}$ and $\mathbf{G}$ be f.g. graded R-modules with minimal graded free resolutions:

$\mu : \mathbf{E} \to \mathbf{G}$ morphism induces maps $\mu^i : E^i \to G^i$, det. up to chain homotopy.

($\mathbf{E}$ and $\mathbf{G}$ lin presented up to order $j \Rightarrow$ the μ^i's are uniquely det. for $i \leq j-1$.)

$$\begin{tikzcd}
\cdots \arrow[r] & E^1 \arrow[r, "e_1"] \arrow[d, "\mu^1"] & E^0 \arrow[r, "e_0"] \arrow[d, "\mu^0"] & \mathbf{E} \arrow[d, "\mu"] \\
\cdots \arrow[r] & G^1 \arrow[r, "g_1"'] & G^0 \arrow[r, "g_0"'] & \mathbf{G}
\end{tikzcd}$$

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$$\begin{tikzcd}
\cdots \arrow[r] & ? \arrow[r] & ? \arrow[r] & \mathbf{F} \arrow[d, hook] \\
\cdots \arrow[r] & E^1 \arrow[r, "e_1"] \arrow[d, "\mu^1"] & E^0 \arrow[r, "e_0"] \arrow[d, "\mu^0"] & \mathbf{E} \arrow[d, "\mu"] \\
\cdots \arrow[r] & G^1 \arrow[r, "g_1"'] & G^0 \arrow[r, "g_0"'] & \mathbf{G}
\end{tikzcd}$$

What can we say about the resolution of the kernel $\mathbf{F}$?

Cutting off pieces of a linear resolution

Theorem A. (B.-Faenzi-Lella)

E and **G**: m-lin presented R-modules, resp. up to order 1 and 2

$\mu : \mathbf{E} \twoheadrightarrow \mathbf{G}$ a surjective morphism, and μ^i's the induced maps

Then $\mathbf{F} = \ker(\mu)$ is generated in deg m and $m+1$, and moreover:

1. if μ^1 is surjective, **F** is generated in deg m and has linear and quadratic syzygies, and $\beta_{0,m}(\mathbf{F}) = \beta_{0,m}(\mathbf{E}) - \beta_{0,m}(\mathbf{G})$;
2. if moreover μ^2 is surjective, **F** is linearly presented and $\beta_{1,m+1}(\mathbf{F}) = \beta_{1,m+1}(\mathbf{E}) - \beta_{1,m+1}(\mathbf{G})$.

Cutting off pieces of a linear resolution

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2.

$$\begin{tikzcd}
 & \cdots \arrow[r] & R(-m-1)^{\alpha_1-\gamma_1} \arrow[r] \arrow[d] & R(-m)^{\alpha_0-\gamma_0} \arrow[r] \arrow[d] & \mathbf{F} \arrow[d, hook] \\
\cdots \arrow[r] & R(-m-2)^{\alpha_2} \arrow[r] \arrow[d, two heads, "\mu^2"] & R(-m-1)^{\alpha_1} \arrow[r] \arrow[d, two heads, "\mu^1"] & R(-m)^{\alpha_0} \arrow[r] \arrow[d, two heads, "\mu^0"] & \mathbf{E} \arrow[d, two heads, "\mu"] \\
\cdots \arrow[r] & R(-m-2)^{\gamma_2} \arrow[r] & R(-m-1)^{\gamma_1} \arrow[r] & R(-m)^{\gamma_0} \arrow[r] & \mathbf{G}
\end{tikzcd}$$

Looking for constant rank matrices

Our goal is to construct **constant rank** matrices from v.b.
What happens if the sheaves $E = \tilde{\mathbf{E}}$ and $G = \tilde{\mathbf{G}}$ are v.b.?

Theorem B. (B.-Faenzi-Lella)
In the assumptions of Theorem A part 1, suppose also that:

(i) $E = \tilde{\mathbf{E}}$ and $G = \tilde{\mathbf{G}}$ are v.b. on $\mathbb{P}^n$ of rank r and s respectively;
(ii) some extra "technical condition" holds.

Set $a = \beta_{0,m}(\mathbf{E}) - \beta_{0,m}(\mathbf{G})$ and $b = \beta_{1,m+1}(\mathbf{E}) - \beta_{1,m+1}(\mathbf{G})$.
Then the presentation matrix V of $\mathbf{F} = \ker(\mu)$ has a linear part of size $a \times b$ and constant corank $r - s$.
Moreover $F = \tilde{\mathbf{F}}$ is isomorphic to the kernel of $\tilde{\mu} : E \to G$.

Remark. μ^2 surjective $\underset{\not\Leftarrow}{\Rightarrow}$ technical condition

(A lot of) explicit examples!

In the applications we consider the case when $G = 0$, i.e. $\mathbf{G}$ Artinian module, so in particular $F = \tilde{\mathbf{F}} \simeq \tilde{\mathbf{E}} = E$.

What do we get?

A veritable **factory of examples** of constant rank matrices!

Why is it good?

- ⋆ can implement the method on a computer (Macaulay2 packages available online)
- ⋆ explicit examples in several cases

Why is it better than previously existing methods?

- ⋆ can avoid cumbersome computations
- ⋆ the method goes beyond projection method and works for small corank!

It all comes together

Let's go back to:

$$0 \to K(t) \to \mathcal{O}_{\mathbb{P}^n}(-m-1+t)^b \to \mathcal{O}_{\mathbb{P}^n}(-m+t)^a \to E(t) \to 0.$$

We saw that an exact sequence of this type corresponds to an element $\eta \in \mathrm{Ext}^2(E, K)$, which via cup product induces maps

$$\mu_t : \mathrm{H}^0(E(t)) \longrightarrow \mathrm{H}^2(K(t)).$$

$\rightsquigarrow$ cup product with η gives: $\mu = \oplus_t \mu_t : \mathrm{H}^0_*(E) \to \mathrm{H}^2_*(K)$.

This is a morphism, homogeneous of degree 0.

For $n \geq 3$ both cohomology groups $\mathrm{H}^1(\mathcal{O}_{\mathbb{P}^n}(-m+t))$ and $\mathrm{H}^2(\mathcal{O}_{\mathbb{P}^n}(-m-1+t))$ vanish for all t, so μ is surjective.

So we are in a good position to apply Theorem B!

It all comes together

Set $\mathbf{E} = \mathrm{H}^0_*(E)$ and $\mathbf{M} = \mathrm{H}^2_*(K)$,
and define Φ as the linear map induced by the μ_t's:

$$\Phi : \mathrm{Ext}^2(E, K) \longrightarrow \mathrm{Hom}_R(\mathbf{E}, \mathbf{M})_0$$

Theorem I, revisited.

Let $n \geq 3$ and $V : R(-m-1)^b \to R(-m)^a$ skew-symmetrizable of constant rank. Set $K = \ker V$ and $E = \mathrm{Coker}\, V$.
Then $K \simeq E^*(-2m-1)$, and $\exists\ \eta \in \mathrm{H}^2(S^2 E^*(-2m-1))$ under the canonical decomposition

$$\mathrm{Ext}^2(E, E^*(-2m-1)) \simeq \mathrm{H}^2(S^2 E^*(-2m-1)) \oplus \mathrm{H}^2(\wedge^2 E^*(-2m-1))$$

such that V presents $\ker \Phi(\eta)$.

Conversely, if $\eta \in \mathrm{H}^2(S^2 E^*(-2m-1))$, $\mu = \Phi(\eta)$ satisfies Theorem B, and $\ker V \simeq E^*(-2m-1)$, V is skew-symmetrizable.

One explicit example

Theorem II, revisited.

When E is a 2-instanton bundle or a generic 4-instanton, the map $\Phi : \mathrm{Ext}^2(E, E^*(-2m-1)) \longrightarrow \mathrm{Hom}_R(\mathbf{E}, \mathbf{M})_0$ is a surjection, and the assumptions of Theorem B are satisfied.

Example. Let's construct a 10×10 skew-symm matrix of linear forms in 4 variables with constant rank 8, starting from a 2-instanton on $\mathbb{P}^3$.

Remark. there used to be **only one example** of this type of spaces, in [Westwick '96].