Calabi–Yau quotients of irreducible hyperkähler 4-folds

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Outline

Notations

Definition

Le Y be a compact Kähler manifold of dimension n. Then Y is called Calabi–Yau variety if

- $K_Y = 0$;
- $h^{i,0}(Y) = 0 \ \forall i = 1, \ldots, n-1$

Definition

Let X be a compact smooth Kähler manifold. Then X is called IHS manifold if

- $\pi_1(X) = 0$;
- $H^{2,0}(X) = \mathbb{C}\omega_X$, where ω_X is a symplectic form.

$X \sim S^{[2]}$ is said *of* $K3^{[2]}$*-type*.

Our question

Classically known:

Let S be a $K3$ surface;

- $\sigma \in \mathrm{Aut}(S)$ symplectic $\Rightarrow$ a minimal resolution of S/σ is a $K3$ surface.
- $\sigma \in \mathrm{Aut}(S)$ non-symplectic $\Rightarrow$ S/σ is Enriques or has $\kappa = -\infty$.

What about higher dimensions?

- Fujiki: X IHS, $\sigma \in \mathrm{Aut}(X)$ symplectic $\Rightarrow$ in general, no resolution of X/σ is IHS
- Boissière-Nieper-Sarti and Oguiso-Schroer: σ non-symplectic fixed-point free $\Rightarrow$ X/σ generalized Enriques variety

Question

What about other quotients X/σ with X IHS and $\sigma \in \mathrm{Aut}(X)$ non-symplectic?

Our question II

Our question

When do we obtain Calabi–Yau manifolds as resolutions of quotients of IHS manifolds?

O'Grady's example:
A general $\langle 2\rangle$-polarized 4-fold X of $K3^{[2]}$-type is a smooth double cover of an EPW sextic $Y_A \subset \mathbb{P}^5$, which is singular along a surface Σ Blowing-up Σ one gets a smooth Calabi–Yau 4-fold Y.
Main features of the example:

1. the covering involution $\iota \in \mathrm{Aut}(X)$ is non-symplectic;
2. the fixed locus X^ι is a Lagrangian surface Σ', whose image is Σ;
3. Σ are all A_1 singularities and can be resolved via a crepant resolution.

A more precise question

When does the quotient X/σ admit a crepant resolution which is a Calabi–Yau?

Main theorem

Theorem

Take:

- X *an IHS of dimension* $2n$;
- $\sigma \in \mathrm{Aut}(X)$ *non-symplectic of order* p *with* $X^\sigma \neq \emptyset$;
- A *the matrix which linearizes* σ *near a component of* X^σ;
- $(\zeta_p^{a_1}, \ldots \zeta_p^{a_{2n}})$, *with* $0 \leq a_i < p$, *the eigenvalues of* A.

Assume $a_{2i-1} + a_{2i} \equiv 1 \mod p$ *for every* $i = 1, \ldots, n$. *Then:*

1. σ *preserves the volume form if and only if* $p|n$.
2. *The singularities of* X/σ *are canonical if and only if* $p = n$ *and all the components of* X^σ *have dimension* $p = n$.

Corollary

If X *is a* $2p$*-dimensional IHS variety and* $\sigma \in \mathrm{Aut}(X)$ *is non-symplectic of order* p *such that all the components of* X^σ *have dimension* p*, then there exists a crepant resolution* Y *of* X/σ *which is a Calabi–Yau* $2p$*-fold.*

Idea of the proof

1. Take $\Omega \in H^{2n,0}(X)$ a volume form. Then $\exists k \in \mathbb{C}^*$ such that

$$\Omega := k\omega_X^n = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{2n}$$

   σ non-symplectic $\Rightarrow \sigma^*\Omega = \zeta_p^n\Omega$.
   Hence: $\sigma^*\Omega = \Omega \Leftrightarrow p|n$.

2. Take $C \subset X^\sigma$ component of codimension > 1 and $\pi : X \to X/\sigma$.
   Then $\mathrm{Sing}(X/\sigma) \supset \pi(C)$.
   Singularities in $\pi(C)$ are canonical $\Leftrightarrow (\sum_{j=1}^{2n} a_j)/p = 1$.

   - $(\sum_{j=1}^{2n} a_j)/p = 1 \Rightarrow n = p$ and $a_j = 0\ \forall j = 1, \ldots, p$
     $\Rightarrow A = (1, \ldots, 1, \zeta_p, \ldots, \zeta_p) \Rightarrow \operatorname{codim} X^\sigma = p$.
   - $\operatorname{codim} X^\sigma = p \Rightarrow (\sum_{j=1}^{2n} a_j)/p = 1$ is easy.

   Canonical singularities $\Rightarrow \exists \beta' : Y \to X/\sigma$ crepant resolution.
   Moreover, $h^{i,0}(Y) = 0\ \forall\, 0 < i < 2p$. $\square$

The crepant resolution

The following diagram commutes

$$\begin{tikzcd} X \arrow[d, "\pi"'] & \widetilde{X} \arrow[l, "\beta"'] \arrow[d, "\pi'"'] \\ X/\sigma & \widetilde{X}/\widetilde{\sigma} = Y \arrow[l, "\beta'"'] \end{tikzcd}$$

with:

- β the blow-up of X along X^σ;
- $\widetilde{\sigma} \in \mathrm{Aut}(\widetilde{X})$ induced by σ;
- π and π' are the quotient maps;
- β' is the blow-up of X/σ in its singular locus.

The case $p = 2$

Assume $p = 2$: we consider non-symplectic involutions on a 4-fold X.

Classification

- Nikulin: non-symplectic involutions on $K3$ surfaces;
- Beauville: topological classification in the $K3^{[2]}$-type case;
- Boissière, C., Sarti: finer lattice-theoretical classification in the $K3^{[2]}$-type case;
- Mongardi, Tari, Wandel: classification in the generalized Kummer 4-folds case.

Beauville: a non-symplectic involution ι on an IHS 4-fold fixes Lagrangian surfaces.

Notation:
- X an IHS 4-fold;
- ι a non-symplectic involution.

Hodge numbers

Suppose $X^\iota = \coprod B_j$. Denote:

- $b := h^0(\coprod B_j)$;
- $c := \sum_{j=1}^{b}(h^{1,0}(B_j))$;
- $d := \sum_{j=1}^{b}(h^{2,0}(B_j))$;
- $e := \sum_{j=1}^{b}(h^{1,1}(B_j))$;

- $t_{1,1} := \dim H^2(X, \mathbb{C})^\iota = \dim H^{1,1}(X)^\iota$;
- $t_{2,1} := \dim H^{2,1}(X)^\iota = \dim H^3(X, \mathbb{C})^\iota$;
- $t_{3,1} := \dim H^{3,1}(X)^\iota$;
- $t_{2,2} := \dim H^{2,2}(X)^\iota$.

Remark: $\dim H^4(X, \mathbb{C})^\iota = 2 + 2t_{3,1} + t_{2,2}$.

Proposition

The Hodge diamond of Y is given by

$$\begin{array}{ll} h^{0,0}(Y) = h^{4,0} = 1, & h^{1,0}(Y) = h^{2,0} = h^{3,0} = 0, \\ h^{1,1}(Y) = t_{1,1} + b, & h^{2,1}(Y) = t_{2,1} + c, \\ h^{2,2}(Y) = t_{2,2} + e, & h^{3,1}(Y) = t_{3,1} + d. \end{array}$$

Proof and consequences

Idea of the proof: $H^{*,*}(Y) = H^{*,*}(\widetilde{X})^{\widetilde{\iota}}$.
Take E the exceptional divisor of the blow up $\beta \Rightarrow E \subset \widetilde{X}^{\widetilde{\iota}}$.
Hence $\dim\left(H^{p,q}(\widetilde{X})^{\widetilde{\iota}}\right) = \dim\left(H^{p,q}(X)^{\iota}\right) + h^{p-1,q-1}(E)$. $\square$

Corollary

Take X a 4-fold of $K3^{[2]}$-type and $\iota \in \mathrm{Aut}(X)$ a non-symplectic involution.
Then any crepant resolution of X/ι is a Calabi–Yau variety with Hodge numbers:

$$
\begin{array}{ll}
h^{0,0} = h^{4,0} = 1, & h^{1,0} = h^{2,0} = h^{3,0} = 0, \\
h^{1,1} = (112 - 19t_{1,1} + 2c - 2d + t_{1,1}^2)/2, & h^{2,1} = c, \\
h^{2,2} = 352 + 2t_{1,1}^2 - 42t_{1,1} + 2c, & h^{3,1} = 21 - t_{1,1} + d.
\end{array}
$$

Natural involutions

$K3$ surfaces

Take S a $K3$ surface and ι_S a non-symplectic involution.

Nikulin: S^{ι_S} consists of N disjoint curves D_j.
Set $N' = \sum_{j=1}^{N} g(D_j)$.

In most cases: $S^{\iota_S} = C \cup \coprod_{j=1}^{N-1} R_j$ with C of genus $g \geq 0$ and R_j all rational curves.

$(S^{[2]})^{\iota_S^{[2]}}$ consists of the following surfaces:

- $C^{[2]}$;
- $N-1$ surfaces isomorphic to $C \times R_j \simeq C \times \mathbb{P}^1$;
- $N-1$ surfaces isomorphic to $(\mathbb{P}^1)^{[2]}$;
- $(N-2)(N-1)/2$ surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$;
- S/ι_S.

The Borcea-Voisin 4-fold

Cynk–Hulek: Take B_1, B_2 two Calabi–Yau's of dimension n_1 and n_2. Take $\iota_i \in \mathrm{Aut}(B_i) \Rightarrow \iota_1 \times \iota_2 \in \mathrm{Aut}(B_1 \times B_2)$.

Suppose:
- ι_i does not preserve the period of B_i;
- all the components of $(B_1 \times B_2)^{(\iota_1 \times \iota_2)}$ have codimension 2.

Then there exists a crepant resolution of $(B_1 \times B_2)/(\iota_1 \times \iota_2)$ which is a Calabi–Yau of dimension $(n_1 + n_2)$, called *of Borcea–Voisin type*.

Here: choose $B_1 = B_2 = S$ and $\iota_1 = \iota_2 = \iota_S \Rightarrow$ any crepant resolution $\beta : Z \to S^2/\iota_S^2$ is a smooth Calabi–Yau 4-fold.

Hodge numbers of Y

$$h^{1,1} = (24 + 3N - 2N' + N^2)/2,$$
$$h^{2,1} = NN',$$
$$h^{3,1} = (20 - 2N + N' + N'^2)/2,$$
$$h^{2,2} = 2(66 + N - N' + N^2 - NN' + N'^2).$$

Hodge numbers of Z

$$h^{1,1} = 20 + 2N - 2N' + N^2,$$
$$h^{2,1} = 2NN',$$
$$h^{3,1} = 20 - 2N + 2N' + N'^2,$$
$$h^{2,2} = 204 + 4N^2 - 4NN' + 4N'^2.$$

Comparison

$$
\begin{tikzcd}
 & & & S \times S \arrow[dll, "\pi"'] \arrow[dd] \arrow[drr] & & & \\
Z_S \arrow[r, dashed, "\beta_Z"] \arrow[dr] & (S \times S)/(\iota_S \times \iota_S) \arrow[drr] & & & & (S \times S)/\sigma \arrow[dll] & S^{[2]} \arrow[l, dashed] \arrow[dl] \\
 & Z_S/\sigma_Z \arrow[rr, dashed] & & (S \times S)/\langle \sigma, \iota_S \times \iota_S \rangle & & S^{[2]}/\iota_S^{[2]} \arrow[ll, dashed] & Y_S \arrow[l, dashed]
\end{tikzcd}
$$

Proposition

1. *The quotient 4-fold* Z_S/σ *is birational to* Y_S.
2. $Z^\sigma = \Sigma_1 \cup \Sigma_2$ *disjoint surfaces.*
3. $\exists\ \widetilde{Z_S/\sigma}$ *a crepant resolution which is a smooth Calabi–Yau birational to* Y_S.

Remarks

Complex deformations

- Deformations of (S, ι_S): $\dim H^{1,1}(S)^{\iota_S} = 10 - N + N'$;
- deformations of Z_S: $h^{3,1}(Z_S) = 20 - 2N + 2N' + N'^2$;
- deformations of $(S^{[2]}, \iota^{[2]})$: $\dim H^{1,1}(S^{[2]})^{\iota^{[2]}} = 11 - N + N'$;
- deformations of Y_S: $h^{3,1}(Y_S) = (20 - 2N + N' + N'^2)/2$.

Case $N' = 0$: all deformations of Z_S come from products of deformations of (S, ι_S).
Every deformation of Y_S is dominated by a deformation of Z_S.

Mirror symmetry

1. If S and $\check{S}$ are mirror $K3$s, then Z_S and $Z_{\check{S}}$ are mirror Calabi–Yau 4-folds, i.e. $h^{1,1}(Z_S) = h^{3,1}(Z_{\check{S}})$ and $h^{2,2}(Z_S) = h^{2,2}(Z_{\check{S}})$.
2. No mirror symmetry on Y_S induced by hyperkähler mirror symmetry.

Quotients of $S \times S$

Take:

- S general $K3$ surface s.t. $\mathrm{Aut}(S) = \langle \iota_S \rangle$;
- $W := S/\iota_S$.

Fact (Oguiso): $\mathrm{Aut}(S \times S) = \langle \iota_S \times \mathrm{id}, \sigma \rangle \cong \mathcal{D}_8$.

Remark

$(S \times S)/\langle \iota_S \times \mathrm{id}, \sigma \rangle \simeq W^{(2)}$

$$
\begin{tikzcd}
 & & & & S \times S \arrow[dllll, "\gamma_1 := /g_2"'] \arrow[dll, "/g_8"] \arrow[d, "\alpha_1 := /g_7"'] \arrow[drr, "/g_5"'] \arrow[drrrr, "/g_3"] & & & & \\
S^{(2)} \arrow[drr, "\gamma_2 := /\overline{g_7}"'] & & (S \times S)/(\sigma \circ (\iota_S \times \iota_S)) \arrow[d, "/\overline{g_2}"'] & & (S \times S)/(\iota_S \times \iota_S) \arrow[dll, "\beta_2 := /\overline{g_2}"] \arrow[drr, "\alpha_2 := /\overline{g_3}"'] & & S \times W \arrow[d, "/\overline{g_7}"'] & & W \times S \arrow[dll, "/\overline{g_7}"'] \\
 & & S^{(2)}/\iota_S^{(2)} \arrow[drr, "\beta_3 := /\overline{g_3}"'] & & & & W \times W \arrow[dll, "\alpha_3 := /\overline{g_2}"'] & & \\
 & & & & W^{(2)} & & & &
\end{tikzcd}
$$

Skip next

Double covers I

Suppose $S^{\iota} = C$ a smooth curve.

Sequence of double covers I

$S \times S \xrightarrow{\alpha_1} (S \times S)/(\iota \times \iota) \xrightarrow{\alpha_2} W \times W \xrightarrow{\alpha_3} W^{(2)}$

Branch points:

α_3: $\mathrm{Sing}(W^{(2)}) = \pi(\Delta_W) \cong W$;

α_2: $\alpha_2(\alpha_1(C \times S)) \cup \alpha_2(\alpha_1(S \times C))$, intersecting in $C \times C$;

α_1: $\mathrm{Sing}((S \times S)/(\iota_S \times \iota_S)) = C \times C$.

Sequence of double covers II

$S \times S \xrightarrow{\alpha_1} (S \times S)/(\iota \times \iota) \xrightarrow{\beta_2} S^{(2)}/\iota^{(2)} \xrightarrow{\beta_3} W^{(2)}$

Branch points:

β_3: $T := \pi(W \times C \cup C \times W)$, singular in $\pi(C \times C)$, intersecting $\mathrm{Sing}(W^{(2)})$ in $\pi(\Delta_C)$;

β_2: $A_1 \cup A_3$, where $\mathrm{Sing}(S^{(2)}/\iota^{(2)}) = A_1 \cup A_2 \cup A_3$ with $A_1 \cong A_2 \cong W$, $A_3 \cong C^{(2)}$ and $\cap A_i = \pi(\Delta_C)$);

α_1: $\mathrm{Sing}((S \times S)/(\iota_S \times \iota_S)) = C \times C$.

Double covers II

Sequence of double covers III

$S \times S \overset{\gamma_1}{\to} S^{(2)} \overset{\gamma_2}{\to} S^{(2)}/\iota^{(2)} \overset{\beta_3}{\to} W^{(2)}$

Branch points:

β_3: $T := \pi(W \times C \cup C \times W)$, singular in $\pi(C \times C)$, intersecting $\mathrm{Sing}(W^{(2)})$ in $\pi(\Delta_C)$;

γ_2: $A_1 \cup A_2$;

γ_1: $\mathrm{Sing}(S^{(2)}) = \pi(\Delta_S)$.

Enriques involutions

Proposition

1. *If ι_S is an Enriques involution on S, then $Y_S \cong \widetilde{Z_S/\sigma_Z}$.*
2. *They are the blow-up of the non ramified double cover of $W^{(2)}$ in its singular locus.*

We can reconstruct both these processes on $S \times S$:

1. take $\widetilde{S \times S} = \mathrm{Bl}_{\Delta_S \cup \Gamma_{\iota_S}}(S \times S)$;
2. take $\widetilde{\iota_S \times \iota_S}, \widetilde{\sigma} \in \mathrm{Aut}(\widetilde{S \times S})$ induced by $\iota_S \times \iota_S$ and σ.

$\Rightarrow \widetilde{S \times S}/\langle \widetilde{\iota_S \times \iota_S}, \widetilde{\sigma} \rangle$ is smooth and in fact isomorphic both to Y_S and to $\widetilde{Z_S/\sigma_S}$.

Projective models

Projective models of general (S, ι_S)

- $\mathrm{NS}(S) = \langle 2 \rangle$, $W \cong \mathbb{P}^2$, $g(C) = 10$;
- $\mathrm{NS}(S) = U(2)$, $W \cong \mathbb{P}^1 \times \mathbb{P}^1$, $g(C) = 9$;
- $\mathrm{NS}(S) = U$, $W \cong \mathbb{F}^4$, ι is induced by the hyperelliptic involution on each fibre of $S \to \mathbb{P}^1$, elliptic fibration with a section.

Work in progress: take $H \in \mathrm{NS}(S)$ such that $\phi_{|H|} : S \to W \subset \mathbb{P}^N$.
Then $\mathcal{O}(H) \boxtimes \mathcal{O}(H)$ induces divisors $D_{S^{(2)}}$, D_Z and D_Y such that:

- $\phi_{|D_{S^{(2)}}|} : S^{(2)} \to W^{(2)}$;
- $\phi_{|D_Z|} : Z \to W \times W$;
- $\phi_{|D_Z|} : Z \to W^{(2)}$.

Conclusions

Open questions:

1. can one construct different (small) resolutions in other cases?
2. find explicit geometric constructions in higher dimensions/higher orders.

Thank you!