On Gushel–Mukai Varieties

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Definition of GM varieties

A Gushel–Mukai variety of dimension n is

- (ordinary type) a transverse intersection

$$X := G(2, V_5) \cap \mathbf{P}(W_{n+5}) \cap Q \subset \mathbf{P}(\textstyle\bigwedge^2 V_5)$$

- (special type) a double cover

$$X \longrightarrow M := G(2, V_5) \cap \mathbf{P}(W_{n+4}) \subset \mathbf{P}(\textstyle\bigwedge^2 V_5)$$

  branched along $M \cap Q$.

Characterization of GM varieties

One has $K_X \underset{\mathrm{lin}}{\equiv} (2-n)H$ and $H^n = 10$. GM manifolds are:

- $(\mathbf{n}=\mathbf{1})$ genus-6 curves with Clifford index 2 (not hyperelliptic, not trigonal, not a plane quintic), special $\Leftrightarrow$ bielliptic;
- $(\mathbf{n}=\mathbf{2})$ degree-10 K3 surfaces whose polarization contains a genus-6 curve with Clifford index 2, special $\Leftrightarrow$ hyperelliptic;
- $(\mathbf{n}=\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6})$ prime Fano n-folds of degree 10, coindex 3;
- $(\mathbf{n}=\mathbf{6})$ only special case occurs.

Hodge structures of GM manifolds

The integral cohomology is torsion-free and the upper Hodge diamonds are

$(n=1)$
$$\begin{array}{ccc} & 1 & \\ 6 & & 6 \end{array}$$

$(n=2)$
$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \end{array}$$

$(n=3)$
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 10 & & 10 & & 0 \end{array}$$

$(n=4)$
$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 22 & & 1 & & 0 \end{array}$$

$(n=5)$
$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 0 & & 0 & & & & \\ & & & 0 & & 1 & & 0 & & & \\ & & 0 & & 0 & & 0 & & 0 & & \\ & 0 & & 0 & & 2 & & 0 & & 0 & \\ 0 & & 0 & & 10 & & 10 & & 0 & & 0 \end{array}$$

$(n=6)$
$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 0 & & 0 & & & & & \\ & & & & 0 & & 1 & & 0 & & & & \\ & & & 0 & & 0 & & 0 & & 0 & & & \\ & & 0 & & 0 & & 2 & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\ 0 & & 0 & & 1 & & 22 & & 1 & & 0 & & 0 \end{array}$$

Period maps for GM manifolds

When n is odd, the Hodge structure has level 1 and there are (p.p.) intermediate Jacobians and period maps

$$\wp_1 \colon \mathscr{M}_1 \longrightarrow \mathscr{A}_6 \quad , \quad \wp_3 \colon \mathscr{M}_3 \longrightarrow \mathscr{A}_{10} \quad , \quad \wp_5 \colon \mathscr{M}_5 \longrightarrow \mathscr{A}_{10}.$$

When n is even, the Hodge structure is of K3 type and the vanishing cohomology defines period maps

$$\wp_2 \colon \mathscr{M}_2 \longrightarrow \mathscr{D}_{19} \quad , \quad \wp_4 \colon \mathscr{M}_4 \longrightarrow \mathscr{D}_{20} \quad , \quad \wp_6 \colon \mathscr{M}_6 \longrightarrow \mathscr{D}_{20},$$

where $\mathscr{D}_m$ is the quasi-projective quotient of a bounded symmetric domain of dimension m by a discrete group of automorphisms.

We know that $\wp_1$ and $\wp_2$ are closed embeddings. What about the other maps?

Period maps for GM manifolds

The dimensions are

n	1	2	3	4	5	6
$\dim(\mathscr{M}_n)$	15	19	22	24	25	25
$\dim$ (period space)	21	19	55	20	55	20

We will show that $\wp_n$ is not injective for $n \geq 3$ and describe its fibers.

The magic trick (O'Grady, Iliev–Manivel)

Given a smooth (ordinary) GM n-fold

$$X := G(2, V_5) \cap \mathbf{P}(W_{n+5}) \cap Q \subset \mathbf{P}(\bigwedge^2 V_5),$$

one can consider and construct

- the space V_6 of quadrics in $\mathbf{P}(W_{n+5})$ containing X;
- the hyperplane $V_5 \subset V_6$ of "Plücker quadrics";
- a Lagrangian subspace $A \subset \bigwedge^3 V_6$ such that

$$\dim(\bigwedge^3 V_5 \cap A) = 5 - n.$$

Parametrization of GM manifolds

This construction can be reversed and gives, for $n \geq 3$, a bijection (V_6 is a fixed 6-dimensional vector space)

$$\boxed{\begin{array}{c}\text{smooth}\\ \text{ordinary}\\ \text{GM } n\text{-folds}\end{array}} \Big/ \text{isom.} = \boxed{\begin{array}{c}\text{hyperplane } V_5 \subset V_6\\ \text{and}\\ \text{Lagrangian } A \subset \bigwedge^3 V_6\\ \text{with no decomposable}\\ \text{vectors such that}\\ \dim(\bigwedge^3 V_5 \cap A) = 5 - n\end{array}} \Big/ \mathrm{PGL}(V_6)$$

EPW stratification

Given a Lagrangian $A \subset \bigwedge^3 V_6$, we define the (dual) EPW stratification

$$Y^{\ell}_{A^\perp} := \{[V_5] \in \mathbf{P}(V_6^\vee) \mid \dim(\bigwedge^3 V_5 \cap A) = \ell\}.$$

We rewrite the correspondence above as a morphism

$$\boxed{\begin{array}{c}\text{moduli space}\\ \mathscr{M}_n^{\mathrm{ord}}\\ \text{of smooth ordinary}\\ \text{GM } n\text{-folds}\end{array}} \xrightarrow{p_n} \boxed{\begin{array}{c}\text{moduli space } \mathscr{E} \text{ of}\\ \text{Lagrangians } A \subset \bigwedge^3 V_6\\ \text{with no decomposable}\\ \text{vectors}\end{array}}$$

and the fiber of $[A]$ is $Y^{5-n}_{A^\perp}$ (modulo automorphisms).

EPW sextics

Theorem (O'Grady)

If the Lagrangian subspace $A \subset \bigwedge^3 V_6$ contains no decomposable vectors,

- $Y_{A^\perp} := Y_{A^\perp}^{\geq 1} \subset \mathbf{P}(V_6^\vee)$ *is an integral sextic hypersurface;*
- $Y_{A^\perp}^{\geq 2} = \mathrm{Sing}(Y_{A^\perp})$ *is an integral normal surface;*
- $Y_{A^\perp}^{\geq 3}$ *is finite and smooth, empty for A general;*
- $Y_{A^\perp}^{\geq 4}$ *is empty.*

Putting everything together, we have

- $p_5 \colon \mathscr{M}_5^{\mathrm{ord}} \to \mathscr{E}$, fibers are complements of hypersurfaces in $\mathbf{P}^5$ (both $\mathscr{M}_5^{\mathrm{ord}}$ and $\mathscr{E}$ are affine);
- $p_4 \colon \mathscr{M}_4^{\mathrm{ord}} \to \mathscr{E}$, fibers are complements of surfaces in hypersurfaces in $\mathbf{P}^5$;
- $p_3 \colon \mathscr{M}_3^{\mathrm{ord}} \to \mathscr{E}$, fibers are surfaces (minus a finite set).

Ignoring stacky issues, these maps fit together and yield morphisms

$$p_n \colon \mathscr{M}_n = \mathscr{M}_n^{\mathrm{ord}} \sqcup \mathscr{M}_n^{\mathrm{spe}} = \mathscr{M}_n^{\mathrm{ord}} \sqcup \mathscr{M}_{n-1}^{\mathrm{ord}} \longrightarrow \mathscr{E}.$$

Corollary

For each $n \in \{3, 4, 5\}$, there exist non-isotrivial families of smooth GM varieties of dimension n parametrized by a proper curve.

Shafarevich conjecture and hyperbolicity

- The Shafarevich conjecture does not hold for Fano threefolds: the set of $\bar{\mathbf{Q}}$-isomorphism classes of Fano threefolds with Picard number 2, defined over $\mathbf{Q}$ and with good reduction outside $\{2, 29\}$, is infinite (the moduli space of blow ups of any line in a smooth intersection of two quadrics in $\mathbf{P}^5_{\mathbf{Q}}$ contains abelian surfaces; Javanpeykar–Loughran). What about GM manifolds?
- This is related (via the Lang–Vojta conjecture) to hyperbolicity of moduli spaces: do they contain entire curves? In our case, $\mathscr{M}_4$ and $\mathscr{M}_5$ do contain many (proper) rational or elliptic curves ($\mathscr{M}_6$ is affine). This is in contrast with a result of Viehweg–Zuo, which says that this cannot happen with moduli space of smooth projective varieties with *ample* canonical bundle.

Main theorem

Theorem (D.–Iliev–Manivel, D.–Kuznetsov)

The period map $\wp_n$ for GM manifolds of dimension $n \in \{3, 4, 6\}$ factors through $p_n \colon \mathscr{M}_n \to \mathscr{E}$. In particular, it has positive dimensional fibers.

When $n = 3$ ([DIM]): explicit birational isomorphisms between GM threefolds with same A.

Periods of double EPW sextics

There is a canonical double cover $\widetilde{Y}_{A^\perp} \to Y_{A^\perp}$.

Theorem (O'Grady)

If A has no decomposable vectors and $Y_{A^\perp}^{\geq 3} = \varnothing$, the fourfold $\widetilde{Y}_{A^\perp}$ is a HK manifold of $K3^{[2]}$-type.

Theorem (D.–Kuznetsov)

If X is a GM manifold of dimension $n \in \{4, 6\}$, with Lagrangian A, there is an isomorphism of polarized Hodge structures

$$H^n(X; \mathbf{Z})_{00} \cong H^2(\widetilde{Y}_{A^\perp}; \mathbf{Z})_0((-1)^{n/2-1}).$$

Periods of double EPW sextics

Corollary (D.–Kuznetsov)

When $n \in \{4, 6\}$, there is a factorization

$$\wp_n \colon \mathscr{M}_n \xrightarrow{p_n} \mathscr{E} \xrightarrow{\wp} \mathscr{D}_{20},$$

where the period map $\wp$ is an open embedding.

The last statement is a theorem of Verbitsky.

Method of proof

We consider the dual situation $\widetilde{Y}_A \to Y_A \subset \mathbf{P}(V_6)$ and the inverse image $\widetilde{Y}_{A,V_5} \subset \widetilde{Y}_A$ of the hyperplane $V_5 \subset V_6$.

As for cubic fourfolds, the isomorphism in the theorem is given by a correspondence, using

- when $n = 4$, the (smooth) variety of lines in X, a small resolution of $\widetilde{Y}_{A,V_5}$;
- when $n = 6$, the (smooth) variety of σ-planes in X, a $\mathbf{P}^1$-bundle over $\widetilde{Y}_{A,V_5}$.

Birationalities

Theorem (D.–Iliev–Manivel, D.–Kuznetsov)

Any GM manifolds of the same dimension with isomorphic associated Lagrangians are birationally isomorphic.

In particular, the rationality of a GM manifold only depends on its associated Lagrangian, hence on its period point.
When $n = 3$, a general GM manifold is not rational (use intermediate Jacobian).
When $n \in \{5, 6\}$, all GM manifolds are rational.
When $n = 4$, the situation is analogous to that of cubic fourfolds: some rational examples are known, but one expects very general GM fourfolds to be irrational.