

On Gushel–Mukai Varieties

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Definition of GM varieties

A Gushel–Mukai variety of dimension n is

- (ordinary type) a transverse intersection

$$X := G(2, V_5) \cap \mathbf{P}(W_{n+5}) \cap Q \subset \mathbf{P}(\wedge^2 V_5)$$

- (special type) a double cover

$$X \longrightarrow M := G(2, V_5) \cap \mathbf{P}(W_{n+4}) \subset \mathbf{P}(\wedge^2 V_5)$$

branched along $M \cap Q$.

Characterization of GM varieties

One has $K_X \cong_{\text{lin}} (2 - n)H$ and $H^n = 10$. GM manifolds are:

- ($n = 1$) genus-6 curves with Clifford index 2 (not hyperelliptic, not trigonal, not a plane quintic), special \Leftrightarrow bielliptic;
- ($n = 2$) degree-10 K3 surfaces whose polarization contains a genus-6 curve with Clifford index 2, special \Leftrightarrow hyperelliptic;
- ($n = 3, 4, 5, 6$) prime Fano n -folds of degree 10, coindex 3;
- ($n = 6$) only special case occurs.

Hodge structures of GM manifolds

The integral cohomology is torsion-free and the upper Hodge diamonds are

$$\begin{array}{cccc}
 (n=1) & (n=2) & (n=3) & (n=4) \\
 \begin{array}{c} 1 \\ 6 \quad 6 \end{array} & \begin{array}{c} 1 \\ 0 \quad 20 \quad 0 \\ 1 \end{array} & \begin{array}{c} 1 \\ 0 \quad 0 \quad 0 \\ 0 \quad 10 \quad 10 \quad 0 \\ 0 \end{array} & \begin{array}{c} 1 \\ 0 \quad 0 \quad 0 \\ 0 \quad 1 \quad 0 \quad 0 \\ 0 \quad 0 \quad 22 \quad 1 \quad 0 \\ 0 \end{array} \\
 \\
 (n=5) & (n=6) \\
 \begin{array}{c} 1 \\ 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 10 \quad 10 \quad 0 \quad 0 \\ 0 \end{array} & \begin{array}{c} 1 \\ 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \\ 0 \quad 0 \quad 1 \quad 22 \quad 1 \quad 0 \quad 0 \\ 0 \end{array}
 \end{array}$$

Period maps for GM manifolds

When n is odd, the Hodge structure has level 1 and there are (p.p.) intermediate Jacobians and period maps

$$\wp_1: \mathcal{M}_1 \longrightarrow \mathcal{A}_6 \quad , \quad \wp_3: \mathcal{M}_3 \longrightarrow \mathcal{A}_{10} \quad , \quad \wp_5: \mathcal{M}_5 \longrightarrow \mathcal{A}_{10}.$$

When n is even, the Hodge structure is of K3 type and the vanishing cohomology defines period maps

$$\wp_2: \mathcal{M}_2 \longrightarrow \mathcal{D}_{19} \quad , \quad \wp_4: \mathcal{M}_4 \longrightarrow \mathcal{D}_{20} \quad , \quad \wp_6: \mathcal{M}_6 \longrightarrow \mathcal{D}_{20},$$

where \mathcal{D}_m is the quasi-projective quotient of a bounded symmetric domain of dimension m by a discrete group of automorphisms.

We know that \wp_1 and \wp_2 are closed embeddings. What about the other maps?

Period maps for GM manifolds

The dimensions are

n	1	2	3	4	5	6
$\dim(\mathcal{M}_n)$	15	19	22	24	25	25
$\dim(\text{period space})$	21	19	55	20	55	20

We will show that \wp_n is not injective for $n \geq 3$ and describe its fibers.

The magic trick (O’Grady, Iliev–Manivel)

Given a smooth (ordinary) GM n -fold

$$X := G(2, V_5) \cap \mathbf{P}(W_{n+5}) \cap Q \subset \mathbf{P}(\wedge^2 V_5),$$

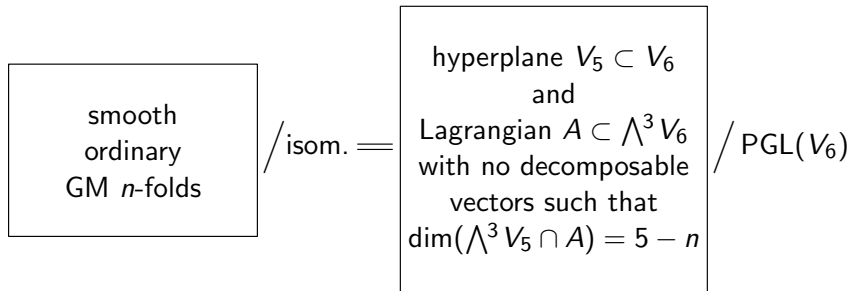
one can consider and construct

- the space V_6 of quadrics in $\mathbf{P}(W_{n+5})$ containing X ;
- the hyperplane $V_5 \subset V_6$ of “Plücker quadrics”;
- a Lagrangian subspace $A \subset \wedge^3 V_6$ such that

$$\dim(\wedge^3 V_5 \cap A) = 5 - n.$$

Parametrization of GM manifolds

This construction can be reversed and gives, for $n \geq 3$, a bijection (V_6 is a fixed 6-dimensional vector space)

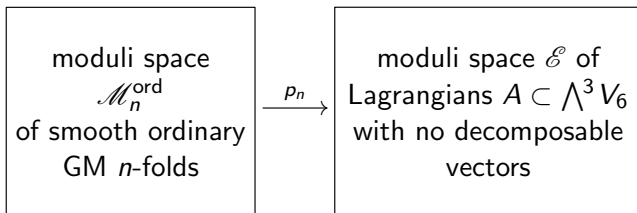


EPW stratification

Given a Lagrangian $A \subset \Lambda^3 V_6$, we define the (dual) EPW stratification

$$Y_{A^\perp}^\ell := \{[V_5] \in \mathbf{P}(V_6^\vee) \mid \dim(\Lambda^3 V_5 \cap A) = \ell\}.$$

We rewrite the correspondence above as a morphism



and the fiber of $[A]$ is $Y_{A^\perp}^{5-n}$ (modulo automorphisms).

EPW sextics

Theorem (O'Grady)

If the Lagrangian subspace $A \subset \wedge^3 V_6$ contains no decomposable vectors,

- $Y_{A^\perp} := Y_{A^\perp}^{\geq 1} \subset \mathbf{P}(V_6^\vee)$ *is an integral sextic hypersurface;*
- $Y_{A^\perp}^{\geq 2} = \text{Sing}(Y_{A^\perp})$ *is an integral normal surface;*
- $Y_{A^\perp}^{\geq 3}$ *is finite and smooth, empty for A general;*
- $Y_{A^\perp}^{\geq 4}$ *is empty.*

Putting everything together, we have

- $p_5: \mathcal{M}_5^{\text{ord}} \rightarrow \mathcal{E}$, fibers are complements of hypersurfaces in \mathbf{P}^5 (both $\mathcal{M}_5^{\text{ord}}$ and \mathcal{E} are affine);
- $p_4: \mathcal{M}_4^{\text{ord}} \rightarrow \mathcal{E}$, fibers are complements of surfaces in hypersurfaces in \mathbf{P}^5 ;
- $p_3: \mathcal{M}_3^{\text{ord}} \rightarrow \mathcal{E}$, fibers are surfaces (minus a finite set).

Ignoring stacky issues, these maps fit together and yield morphisms

$$p_n: \mathcal{M}_n = \mathcal{M}_n^{\text{ord}} \sqcup \mathcal{M}_n^{\text{spe}} = \mathcal{M}_n^{\text{ord}} \sqcup \mathcal{M}_{n-1}^{\text{ord}} \longrightarrow \mathcal{E}.$$

Corollary

For each $n \in \{3, 4, 5\}$, there exist non-isotrivial families of smooth GM varieties of dimension n parametrized by a proper curve.

Shafarevich conjecture and hyperbolicity

- The Shafarevich conjecture does not hold for Fano threefolds: the set of $\bar{\mathbf{Q}}$ -isomorphism classes of Fano threefolds with Picard number 2, defined over \mathbf{Q} and with good reduction outside $\{2, 29\}$, is infinite (the moduli space of blow ups of any line in a smooth intersection of two quadrics in $\mathbf{P}_{\mathbf{Q}}^5$ contains abelian surfaces; Javanpeykar–Loughran). What about GM manifolds?
- This is related (via the Lang–Vojta conjecture) to hyperbolicity of moduli spaces: do they contain entire curves? In our case, \mathcal{M}_4 and \mathcal{M}_5 do contain many (proper) rational or elliptic curves (\mathcal{M}_6 is affine). This is in contrast with a result of Viehweg–Zuo, which says that this cannot happen with moduli space of smooth projective varieties with *ample* canonical bundle.

Main theorem

Theorem (D.–Iliev–Manivel, D.–Kuznetsov)

The period map \wp_n for GM manifolds of dimension $n \in \{3, 4, 6\}$ factors through $p_n: \mathcal{M}_n \rightarrow \mathcal{E}$. In particular, it has positive dimensional fibers.

When $n = 3$ ([DIM]): explicit birational isomorphisms between GM threefolds with same A .

Periods of double EPW sextics

There is a canonical double cover $\tilde{Y}_{A^\perp} \rightarrow Y_{A^\perp}$.

Theorem (O'Grady)

If A has no decomposable vectors and $Y_{A^\perp}^{\geq 3} = \emptyset$, the fourfold \tilde{Y}_{A^\perp} is a HK manifold of $K3^{[2]}$ -type.

Theorem (D.–Kuznetsov)

If X is a GM manifold of dimension $n \in \{4, 6\}$, with Lagrangian A , there is an isomorphism of polarized Hodge structures

$$H^n(X; \mathbf{Z})_{00} \cong H^2(\tilde{Y}_{A^\perp}; \mathbf{Z})_0((-1)^{n/2-1}).$$

Periods of double EPW sextics

Corollary (D.–Kuznetsov)

When $n \in \{4, 6\}$, there is a factorization

$$\wp_n: \mathcal{M}_n \xrightarrow{p_n} \mathcal{E} \xrightarrow{\wp} \mathcal{D}_{20},$$

where the period map \wp is an open embedding.

The last statement is a theorem of Verbitsky.

Method of proof

We consider the dual situation $\tilde{Y}_A \rightarrow Y_A \subset \mathbf{P}(V_6)$ and the inverse image $\tilde{Y}_{A,V_5} \subset \tilde{Y}_A$ of the hyperplane $V_5 \subset V_6$.

As for cubic fourfolds, the isomorphism in the theorem is given by a correspondence, using

- when $n = 4$, the (smooth) variety of lines in X , a small resolution of \tilde{Y}_{A,V_5} ;
- when $n = 6$, the (smooth) variety of σ -planes in X , a \mathbf{P}^1 -bundle over \tilde{Y}_{A,V_5} .

Birationalities

Theorem (D.–Iliev–Manivel, D.–Kuznetsov)

Any GM manifolds of the same dimension with isomorphic associated Lagrangians are birationally isomorphic.

In particular, the rationality of a GM manifold only depends on its associated Lagrangian, hence on its period point.

When $n = 3$, a general GM manifold is not rational (use intermediate Jacobian).

When $n \in \{5, 6\}$, all GM manifolds are rational.

When $n = 4$, the situation is analogous to that of cubic fourfolds: some rational examples are known, but one expects very general GM fourfolds to be irrational.