The CM representation type of varieties

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2. Representation type: some examples and the main result
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2. Representation type: some examples and the main result

3. Proof of the main result
Determinantal representations

\[ f = 2x^3 - x^2 - y^2 - 2x + 1 \in R = \mathbb{R}[x, y] \]
Determinantal representations

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\[
\begin{vmatrix}
1 & x & 0 \\
x & 1 & y \\
0 & y & 1 - 2x
\end{vmatrix}
\]
Determinantal representations

\[ f = x_0 x_1 + x_2 x_3 = \det(M) \]
Determinantal representations

\[ f = x_0 x_1 + x_2 x_3 = \det(M) \]

\[ M = \begin{pmatrix} x_0 & -x_2 \\ x_3 & x_1 \end{pmatrix} \]
Determinantal representations

\[ f = x_0 x_1 + x_2 x_3 + x_4 x_5 + x_6 x_7 \]
Determinantal representations

\[ f = x_0 x_1 + x_2 x_3 + x_4 x_5 + x_6 x_7 \]

\[ f^4 = \begin{vmatrix}
 0 & 0 & 0 & -x_4 & 0 & -x_2 & x_6 & -x_0 \\
 0 & 0 & x_4 & 0 & x_2 & 0 & -x_1 & -x_7 \\
 0 & -x_4 & 0 & 0 & -x_6 & x_1 & 0 & -x_3 \\
 x_4 & 0 & 0 & 0 & x_0 & x_7 & x_3 & 0 \\
 0 & -x_2 & x_6 & -x_1 & 0 & 0 & 0 & x_5 \\
 x_2 & 0 & -x_0 & -x_7 & 0 & 0 & -x_5 & 0 \\
 -x_6 & x_0 & 0 & -x_3 & 0 & x_5 & 0 & 0 \\
 x_1 & x_7 & x_3 & 0 & -x_5 & 0 & 0 & 0 
\end{vmatrix} \]
Given $f \Rightarrow$ construct $M$ with $\det(M) = f^r$, $r = \text{rank}$. 
1. Given $f$ ⇒ construct $M$ with $\det(M) = f^r$, $r = \text{rank}$.
2. Fixed $f$, describe moduli of all $M$. 

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$$M \leftrightarrow \mathcal{E} \text{ sheaf on } X: \mathcal{E}_x = \text{coker}(M_x).$$

Cohen-Macaulay condition $\mathcal{E} \in \text{CM}(X)$:
1. Given $f \Rightarrow$ construct $M$ with $\det(M) = f^r$, $r =$rank.

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Cohen-Macaulay condition $\mathcal{E} \in \text{CM}(X)$:

- Sheaf $\mathcal{E} = \text{coker}(M)$ locally CM, without intermediate cohomology

\[ H^i_*(E) = \bigoplus_t H^i(\mathcal{E}(t)) \text{ vanishes for } 0 < i < n. \]
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Cohen-Macaulay condition $\mathcal{E} \in \text{CM}(X)$:

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$$H_*^i(E) = \bigoplus_t H^i(\mathcal{E}(t)) \text{ vanishes for } 0 < i < n.$$  

- Module $E = \text{coker}(M) = H_*^0(\mathcal{E})$ is MCM on $\mathbb{K}[X]$.  

1. Given $f \implies$ construct $M$ with $\det(M) = f^r$, $r = \text{rank}$. 

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**Cohen-Macaulay condition $\mathcal{E} \in \text{CM}(X)$:**

- Sheaf $\mathcal{E} = \text{coker}(M)$ locally CM, without intermediate cohomology
  $$H^i_\ast(E) = \bigoplus_t H^i(\mathcal{E}(t)) \quad \text{vanishes for } 0 < i < n.$$

- Module $E = \text{coker}(M) = H^0_\ast(\mathcal{E})$ is MCM on $\text{IK}[X]$.

- $R$-resolution of $E$ has length $N - n$. 
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3. Proof of the main result
Representation type of quivers

Quiver $Q$: finite (connected) directed graph. No oriented loops
$\mathbb{K}[Q]$-modules $\text{Rep}(Q)$ category of $\mathbb{K}$-linear maps indexed by arrows of $Q$
Representation type of quivers

Representation type measures the complexity of $\text{Rep}(Q)$

**Finite**

Indecomposables of $\text{Rep}(Q)$ are finitely many up to iso.

- Dynkin diagrams $A$, $D$, $E$.
- Tame: Indecomposables of $\text{Rep}(Q)$ vary in families of dimension 1.
- Extended Dynkin $\tilde{A}$, $\tilde{D}$, $\tilde{E}$.
- Wild: For any $\Lambda$ $K$-algebra with $\dim(\Lambda) < \infty$, $\text{Mod}(\Lambda) \hookrightarrow \text{Rep}(Q)$. Any other quiver.
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\[ \forall \Lambda \text{ } \mathbb{K}\text{-algebra with } \dim(\Lambda) < \infty, \text{Mod}(\Lambda) \hookrightarrow \text{Rep}(Q). \]
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\[
\forall \Lambda \text{ } \text{K-algebra with } \dim(\Lambda) < \infty, \text{ } \text{Mod}(\Lambda) \hookrightarrow \text{Rep}(Q). \text{ Any other quiver.}
\]
To be more precise...

Meanings of $\text{Mod}(\Lambda) \leftrightarrow \text{Rep}(Q)$

- There is a fully faithful functor $\text{Mod}(\Lambda) \rightarrow \text{Rep}(Q)$ (strictly wild)

$\exists$ functor $\Phi : \text{Mod}(\Lambda) \rightarrow \text{Rep}(Q)$ which is a representation embedding

$\Phi(M) \cong \Phi(M')$ iff $M \cong M'$;

$\Phi(M)$ is decomposable iff $M$ is decomposable.
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CM Representation type of varieties

Complexity of the category $\text{CM}(X)$, for $X \subset \mathbb{P}^n$ projective.
CM Representation type of varieties

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**Finite**

Indecomposables of $\text{CM}(X)$ are finitely many up to iso.

- $\mathbb{P}^n$
- smooth quadrics
- rational curves
- $v_2(\mathbb{P}^2)$
- $S(1, 2) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2))$
**CM Representation type of varieties**

Complexity of the category $\text{CM}(X)$, for $X \subset \mathbb{P}^n$ projective.

<table>
<thead>
<tr>
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<table>
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<tr>
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<tbody>
<tr>
<td>Indecomposables of $\text{CM}(X)$ vary in families of bounded dimension.</td>
</tr>
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<td>curve of genus 1 (with ODP)</td>
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<td>$S(2, 2), S(1, 3)$</td>
</tr>
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</tr>
<tr>
<td>quadrics of corank 1</td>
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</table>

Countable or continuous families of dimension 1!
**CM Representation type of varieties**

Complexity of the category CM($X$), for $X \subset \mathbb{P}^n$ projective.

\begin{itemize}
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      \end{itemize}
      \item Countable or continuous families of dimension 1!
    \end{itemize}
  \item **Wild**
    \begin{itemize}
      \item \forall \Lambda \mathbb{K}$-algebra of finite dimension, Mod($\Lambda \hookrightarrow$ CM($X$)).
    \end{itemize}
\end{itemize}
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3 Proof of the main result
Representation type of curves

\[ d = 1 \]  
Finite

\[ d = 2 \]  
Countable

\[ d = 2 \]

Degree \( d \leq 2 \)
Representation type of curves

<table>
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<tr>
<th>Degree $d$</th>
<th>Tame</th>
<th>Wild</th>
</tr>
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<tbody>
<tr>
<td>$d = 3$</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$d \geq 4$</td>
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<td></td>
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Degree $d \geq 3$
### Representation type of curves

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<th>Drozd-Greuel CM type of curves</th>
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<td>- Finite iff rational normal curve</td>
</tr>
<tr>
<td>- Countable iff $p_g = 0$ with ODP</td>
</tr>
<tr>
<td>- Tame iff $p_g = 1$ (perhaps with ODP)</td>
</tr>
<tr>
<td>- Wild iff $p_g \geq 2$ or more singular than ODP</td>
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3. Proof of the main result
## Theorem

Let $X \subset \mathbb{P}^N$ be a closed integral subscheme, $n = \dim(X) > 0$. Assume:

- $\mathbb{K}$ is algebraically closed;
- $X$ is not a cone;
- $\mathcal{O}_X \in \text{CM}(X)$ i.e. $X$ is ACM;
- $X$ is not one of the 9 finite or tame cases we already met.
CM-representation type of ACM varieties

Theorem

Let $X \subset \mathbb{P}^N$ be a closed integral subscheme, $n = \dim(X) > 0$. Assume:

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- $X$ is not one of the 9 finite or tame cases we already met.

Then $X$ is of \textit{wild CM type}.
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Enough to be wild

$\text{Rep}(K_s) \hookrightarrow \text{CM}(X)$, $\exists s \geq 3$

1. Quiver $K_s$ with 2 vertexes and $s$ arrows

![Quiver diagram](image)
Enough to be wild

\textbf{Rep}(K_s) \hookrightarrow \text{CM}(X), \exists s \geq 3

1. Quiver $K_s$ with 2 vertexes and $s$ arrows

\begin{tikzpicture}
  \node (a) at (0,0) {•};
  \node (b) at (1,0) {•};
  \draw[->] (a) to[bend right] (b);
  \draw[->] (b) to[bend right] (a);
\end{tikzpicture}

2. Condition ($\star$):

\[ \dim K \text{Ext}^1_X(E,F) = s \geq 3. \]
Enough to be wild

\[ \text{Rep}(K_s) \leftrightarrow \text{CM}(X), \exists s \geq 3 \]

1. Quiver $K_s$ with 2 vertexes and $s$ arrows

\[ \circ \xrightarrow{\quad} \circ \]

2. Condition (⋆): find simple $\mathcal{E} \perp \mathcal{F} \in \text{CM}(X)$ with

\[
\dim_{\mathbb{K}} \text{Ext}^1_X(\mathcal{E}, \mathcal{F}) = s \geq 3.
\]
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3. Proof of the main result
Linear section $Y \subseteq X$ of dimension $m = n - c$

Key lemma. Set $\text{dim}(Y) = m > 0$. 

Need $K^Y(m - 1)$ effective, equivalent to positive sectional genus.
Linear section $Y \subset X$ of dimension $m = n - c$

Key lemma. Set $\dim(Y) = m > 0$.

1. $\mathcal{E} \in U_r(Y)$ rank-$r$ Ulrich sheaf $\mathcal{E} \in \text{CM}(Y)$ with $R$-linear resolution.
Linear section $Y \subset X$ of dimension $m = n - c$

Key lemma. Set $\dim(Y) = m > 0$.

1. $\mathcal{E} \in \mathcal{U}_r(Y)$ rank-$r$ Ulrich sheaf $\mathcal{E} \in \mathcal{CM}(Y)$ with $R$-linear resolution.
2. $\mathcal{U}(Y)$ category of Ulrich sheaves on $Y$.

$$\Omega^c : \mathcal{U}(Y) \hookrightarrow \mathcal{CM}(X)$$

c-th stable syzygy functor of $E$ as $\mathbb{K}[X]$-module.
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c-th stable syzygy functor of $E$ as $\mathbb{K}[X]$-module.
3. Stable category $\text{CM}(X)$ kills morphisms through free modules.
Linear section \( Y \subset X \) of dimension \( m = n - c \)

**Key lemma.** Set \( \dim(Y) = m > 0 \).

1. \( \mathcal{E} \in \mathcal{U}_r(Y) \) rank-\( r \) Ulrich sheaf \( \mathcal{E} \in \text{CM}(Y) \) with \( R \)-linear resolution.
2. \( \text{U}(Y) \) category of **Ulrich** sheaves on \( Y \).

\[
\Omega^c : \text{U}(Y) \hookrightarrow \text{CM}(X)
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\( c \)-th stable syzygy functor of \( E \) as \( \mathbb{K}[X] \)-module.

3. Stable category \( \text{CM}(X) \) kills morphisms through free modules.
4. Recover \( E^*(c) \) as quotient of \( \Omega^c(E)^* \) by elements of degree \( \leq 1 - c \).
Linear section \( Y \subset X \) of dimension \( m = n - c \)

Key lemma. Set \( \dim(Y) = m > 0 \).

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Minimal degree $d = N - n + 1$

Warm up: del Pezzo-Bertini

- $X$ not a cone $\Rightarrow X$ smooth.
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Warm up: del Pezzo-Bertini

- $X$ not a cone $\Rightarrow$ $X$ smooth.
- $X = v_2(\mathbb{P}^2)$ or $X = S(\bar{a})$ or a quadric.
Minimal degree \( d = N - n + 1 \)

**Warm up: del Pezzo-Bertini**

- \( X \) not a cone \( \Rightarrow \) \( X \) smooth.
- \( X = v_2(\mathbb{P}^2) \) or \( X = S(\vec{a}) \) or a quadric.
- \( E = O_X(H - F) \) and \( F = O_X((d - 1)F) \) give (\( \star \)). \( X \) strictly Ulrich wild.

Except for quadrics, \( v_2(\mathbb{P}^2) \), \( S(1,2) \), \( S(1,3) \) and \( S(2,2) \).

Tame cases

Use derived categories for \( S(1,3) \) and \( S(2,2) \).

Work in progress for singular cases e.g. \( S(0,3) \).
Minimal degree $d = N - n + 1$

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**Tame cases**

- Use derived categories for $S(1, 3)$ and $S(2, 2)$.
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Tame cases

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More on tame cases

With Francesco Malaspina: $\text{CM}(X)$ for $S(1, 3)$ and $S(2, 2)$.

1. Only Ulrich bundles move. Finitely many more.
More on tame cases

With Francesco Malaspina: $\text{CM}(X)$ for $S(1, 3)$ and $S(2, 2)$.

1. Only Ulrich bundles move. Finitely many more.
2. Indecomposables in $U_r(X)$ are a $\mathbb{P}^1$ for $r$ even, a point for $r$ odd.
More on tame cases

With Francesco Malaspina: \( \text{CM}(X) \) for \( S(1, 3) \) and \( S(2, 2) \).

1. Only Ulrich bundles move. Finitely many more.
2. Indecomposables in \( U_r(X) \) are a \( \mathbb{P}^1 \) for \( r \) even, a point for \( r \) odd.
3. Use derived \( \pi^* \pi_* \mathcal{E} \rightarrow \mathcal{E} \) with \( \pi : X \rightarrow \mathbb{P}^1 \) to unwind \( \mathcal{E} \).
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3. Proof of the main result
“High” degree $d \geq N - n + 3$

Reduce to curves, $m = 1$

- $p_g(Y) \geq 2$. 
"High" degree \( d \geq N - n + 3 \)

Reduce to curves, \( m = 1 \)

- \( p_g(Y) \geq 2 \).
- Take \( F \) and \( E \) generic in the compactified Jacobian.
“High” degree $d \geq N - n + 3$

Reduce to curves, $m = 1$

- $p_g(Y) \geq 2$.
- Take $\mathcal{F}$ and $\mathcal{E}$ generic in the compactified Jacobian.
- If $c_1(\mathcal{E}) = c_1(\mathcal{F}) = d + g - 1$ then $\mathcal{E} \perp \mathcal{F} \in U_1(Y)$.
“High” degree $d \geq N - n + 3$

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- Take $\mathcal{F}$ and $\mathcal{E}$ generic in the compactified Jacobian.
- If $c_1(\mathcal{E}) = c_1(\mathcal{F}) = d + g - 1$ then $\mathcal{E} \perp \mathcal{F} \in U_1(Y)$.
- Use extensions to reach $\mathcal{E}' \perp \mathcal{F}' \in U_2(Y)$ with $(\star)$. 
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Almost minimal degree $d = N - n + 2$

Reduce to del Pezzo surfaces, $m = 2$

- $Y$ may be non-normal.
Almost minimal degree $d = N - n + 2$

Reduce to del Pezzo surfaces, $m = 2$

- $Y$ may be non-normal.
- Normal $Y$: $\mathcal{E} \perp \mathcal{F} \in U_2(Y)$ with (★).
  Easy by Serre construction from $d + 2$ points.
Almost minimal degree \( d = N - n + 2 \)

Reduce to del Pezzo surfaces, \( m = 2 \)

- \( Y \) may be non-normal.
- Normal \( Y \): \( \mathcal{E} \perp \mathcal{F} \in U_2(Y) \) with (⋆).
  
  Easy by Serre construction from \( d + 2 \) points.

- Beware \( U_1(Y) = \emptyset \) e.g. if \( Y \) is a cubic with \( E_6 \) singularity.
Almost minimal degree $d = N - n + 2$

Reduce to del Pezzo surfaces, $m = 2$

- $Y$ may be non-normal.
- Normal $Y$: $\mathcal{E} \perp \mathcal{F} \in U_2(Y)$ with $(\star)$.
  Easy by Serre construction from $d + 2$ points.
- Beware $U_1(Y) = \emptyset$ e.g. if $Y$ is a cubic with $E_6$ singularity.
- If $Y$ is not normal then $\bar{Y}$ has minimal degre.
Non normal del Pezzo surfaces

$\mathcal{E} \perp \mathcal{F}$ image of Ulrich line bundles on $\bar{Y}$ via $\pi : \bar{Y} \to Y$.

- E.g. $Y$ cubic surface with a line $L$ of singularities

$$x_1^3 + x_0^2x_2 - x_0x_1x_3 = \begin{vmatrix} x_0 & x_1 & 0 \\ x_1 & x_3 & -x_0 \\ 0 & x_2 & -x_1 \end{vmatrix}$$
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- Use \( A' \) and \( B' \) projected from \( \bar{Y} \) to compute \( \text{Ext}^1_Y(\mathcal{E}, \mathcal{F}) \) and deduce (\( \star \)).
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8. Remove Ulrich sheaves. Does $X$ remain wild?
If $X$ is not ACM? Abelian surfaces are strictly Ulrich wild (Beauville).
Questions – extended version

1. If $X$ is not ACM? Abelian surfaces are strictly Ulrich wild (Beauville).
2. When happens for cones? Most are wild. $S(0, 3)$ should be tame.
Questions – extended version

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2. Whan happens for cones? Most are wild. $S(0, 3)$ should be tame.
3. Whan happens for reducible or non-reduced varieties?

4. $X$ smooth and $r$ fixed. Is the family of ACM sheaves of rank $r$ bounded?

5. Non-projective varieties? Elliptic singularities are tame.


7. Classify rigid ACM sheaves on (some) CM-wild varieties. Done for $v_3(\mathbb{P}^2)$ and $v_2(\mathbb{P}^3)$ (cf. Iyama-Yoshino).

8. $X$ projective, not one of the CM-finite or tame cases. Is $X$ (strictly) Ulrich wild? If we remove Ulrich sheaves, does $X$ remain CM-wild?
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