The CM representation type of varieties

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Plan

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1. Introduction

2. Representation type: some examples and the main result

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3. Proof of the main result

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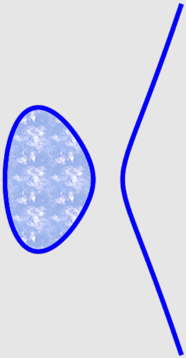
$$f = \begin{vmatrix} 1 & x & 0 \\ x & 1 & y \\ 0 & y & 1-2x \end{vmatrix}$$



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- R-resolution of E has length $N - n$.

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Representation type of quivers

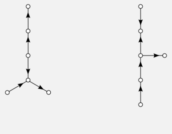
Quiver Q: finite (connected) directed graph. No ortiented loops
$\mathbb{K}[Q]$-modules $\mathrm{Rep}(Q)$ category of $\mathbb{K}$-linear maps indexed by arrows of Q

Representation type of quivers

Representation type measures the complexity of $\mathrm{Rep}(Q)$

Finite

Indecomposables of $\mathrm{Rep}(Q)$ are finitely many up to iso.



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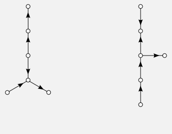

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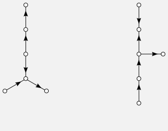

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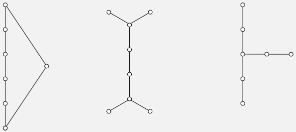

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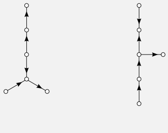

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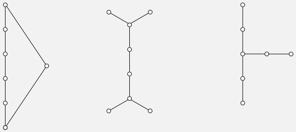

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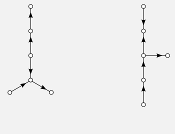

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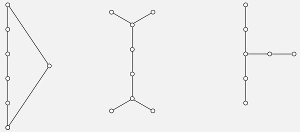

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$\forall \Lambda$ $\mathbb{K}$-algebra with $\dim(\Lambda) < \infty$, $\mathrm{Mod}(\Lambda) \hookrightarrow \mathrm{Rep}(Q)$.

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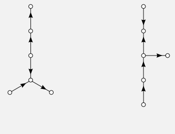

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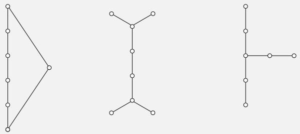

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$\forall \Lambda$ $\mathbb{K}$-algebra with $\dim(\Lambda) < \infty$, $\mathrm{Mod}(\Lambda) \hookrightarrow \mathrm{Rep}(Q)$. *Any other quiver.*

To be more precise…

Meanings of $\mathrm{Mod}(\Lambda) \hookrightarrow \mathrm{Rep}(Q)$

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- $\mathbb{P}^n$
- smooth quadrics
- rational curves
- $v_2(\mathbb{P}^2)$
- $S(1,2) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2))$

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Countable or continuous families of dimension 1!

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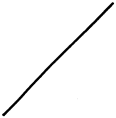
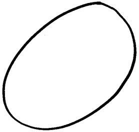


$d = 1$



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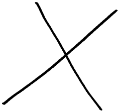
Finite



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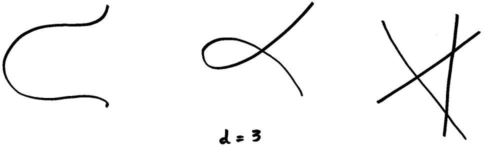
Countable

Degree $d \leq 2$

Representation type of curves



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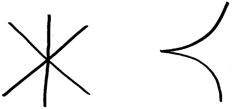

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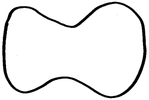

$d \geq 4$

Degree $d \geq 3$

Representation type of curves

Drozd-Greuel CM type of curves

- Finite iff rational normal curve
- Countable iff $p_g = 0$ with ODP
- Tame iff $p_g = 1$ (perhaps with ODP)
- Wild iff $p_g \geq 2$ or more singular than ODP

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CM-representation type of ACM varieties

Theorem

Let $X \subset \mathbb{P}^N$ be a closed integral subscheme, $n = \dim(X) > 0$. Assume:

- *$\mathbb{K}$ is algebraically closed;*
- *X is not a cone;*
- *$\mathcal{O}_X \in \mathrm{CM}(X)$ i.e. X is ACM;*
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Then X is of wild CM type.

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Enough to be wild

$\mathrm{Rep}(K_s) \hookrightarrow \mathrm{CM}(X)$, $\exists s \geq 3$

1. Quiver K_s with 2 vertexes and s arrows

$$\begin{tikzcd} \circ \arrow[r, bend left] \arrow[r] \arrow[r, bend right] & \circ \end{tikzcd}$$

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2. Condition $(\star)$: find simple $\mathcal{E} \perp \mathcal{F} \in \mathrm{CM}(X)$ with

$$\dim_{\mathbb{K}} \mathrm{Ext}^1_X(\mathcal{E}, \mathcal{F}) = s \geq 3.$$

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5. Need $K_Y(m-1)$ effective, equivalent to positive sectional genus.

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Warm up: del Pezzo-Bertini

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- Work in progress for singular cases e.g. $S(0,3)$.

More on tame cases

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1. Only Ulrich bundles move. Finitely many more.
2. Indecomposables in $\mathsf{U}_r(X)$ are a $\mathbb{P}^1$ for r even, a point for r odd.
3. Use derived $\pi^*\pi_*\mathcal{E} \to \mathcal{E}$ with $\pi : X \to \mathbb{P}^1$ to unwind $\mathcal{E}$.

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"High" degree $d \geq N - n + 3$

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- If $c_1(\mathcal{E}) = c_1(\mathcal{F}) = d + g - 1$ then $\mathcal{E} \perp \mathcal{F} \in \mathrm{U}_1(Y)$.
- Use extensions to reach $\mathcal{E}' \perp \mathcal{F}' \in \mathrm{U}_2(Y)$ with $(\star)$.

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Almost minimal degree $d = N - n + 2$

Reduce to del Pezzo surfaces, $m = 2$

- Y may be non-normal.

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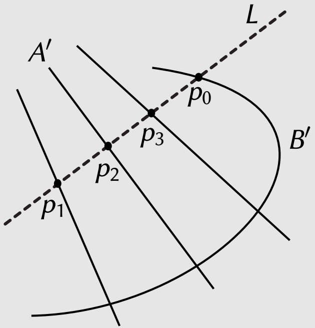

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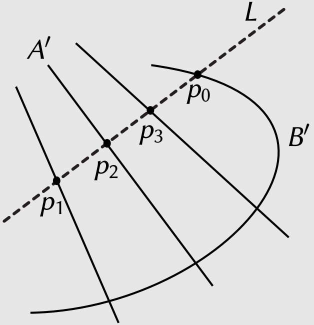

- Use A' and B' projected from $\bar{Y}$ to compute $\mathcal{E}xt^1_Y(\mathcal{E}, \mathcal{F})$ and deduce $(\star)$.

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