Hyperkahler fourfolds and Kummer surfaces

joint work with: A. Iliev, M. Kapustka, K. Ranestad

24th May 2016

1. Known constructions of projective hyper-Kähler (HK) manifolds
2. Double EPW cubes: new projective HK sixfolds
3. Singular double EPW cubes
    - HK fourfolds from singular double EPW cubes
    - Connections to Verra threefolds and cubic fourfolds containing a plane
    - New construction of Kummer surfaces

K3 surfaces

2	4	6	...	22	...	38
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- Kondo, Gritsenko, Hukek, Sankaran proved that the moduli space $\mathcal{M}_{2d}$ of polarised $K3$ surfaces of degree $2d$ with $d > 61$ is of general type.

The table of projective HK

- Let (X, L) be a polarised HK manifolds of dimension $2n$. Then $L^{2n} = c(q(L)^n)$ where c is a constant called the Fujiki constant and $q(.)$ a quadratic form called the Beauville–Bogomolov form (for HK of $K3^{[n]}$ type $c = \frac{(2n)!}{n!2^n}$).

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- The list of all known complete families of projective HK manifolds of dimension $2n > 2$:

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8	[LLSV]			...		...	
6		[IKKR]		...		...	
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- [IR] Iliev, Ranestad: variety of sum of power $VSP(X_3, 10)$,
- [LLSV] Lehn, Lehn, Sorger, van Straten: from Hilbert scheme of twisted cubics on $X_3 \subset \mathbb{P}^5$.

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- Constructions of Kummer quartic surfaces.

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- For general A the variety $D_A^2 \subset G(3, V)$ has dimension 6 and is singular along D_A^3 we call it an *EPW cube*.

double EPW cubes

Theorem (IKKR)

For a general $A \in LG_\eta(10, \wedge^3 V)$ there exists a double cover $X_A \xrightarrow{2:1} D_A^2$ ramified along D_A^3 being a HK sixfold with Beauville degree $q = 4$ (divisibility 2).

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- O'Grady: there exists a double cover of B_A^1 branched along B_A^2 being a projective HK fourfold with $q = 2$.

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Sketch of Proof

- $P(T_p) \cap G(3, V) \subset P(\wedge^3 V)$ is isomorphic to a cone $C_p \subset \mathbb{P}^9$ with vertex p over the Segre $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$.

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- $P(T_p) \cap D_A^3$ is the singular locus of $Y_A = D_A^2 \cap P(T_p)$ and $Z_A \to Y_A$ is a double cover branched along this surface.

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- The first one $U(2)$ is the invariant lattice of $Z_A \to Y_A$.

Z_A and the involutions on HK 4-folds

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related Verra threefolds

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- $(2,2)$ divisors in $\mathbb{P}^2 \times \mathbb{P}^2$ studied by Verra in relation to the counterexample of the tetragonal conjecture about the injectivity of the Prym map.

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Theorem (IKKR)

The Hilbert scheme $\mathcal{H}_A$ of $(1,1)$ conics on F_A admits a $\mathbb{P}^1$ fibration with base Z_A being a HK fourfold with $q = 4$.

Analogy of F_A with cubic containing a plane.

Let S be a general $K3$ surface of degree 2. There are three type of elements $\alpha_1, \alpha_2, \alpha_3$ in $Br(S)_2 = (\mathbb{Z}_2)^{21}$ giving rise to three different Hodge structures of the following manifolds:

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We consider after Yoshioka the moduli spaces $M(S, \alpha_i)$ of twisted sheaves on (S, α_i) for $i = 1, 2, 3$.

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Proposition

$M(S, \alpha_3)$ is isomorphic to the base of a natural $\mathbb{P}^1$–fibration on the Hilbert scheme of $(1,1)$ conics on the corresponding double cover F_A of $\mathbb{P}^2 \times \mathbb{P}^2$ branched along a $(2,2)$ divisor.

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Question

Is the double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ branched along a generic $(2,2)$ divisor rational?

New construction of Kummer surfaces

There exists a quartic hypersurface $K_4 \subset \mathbb{P}^9$ such that the intersection

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Corollary

The Hilbert scheme of $(1,1)$ conics on the threefold being the double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ ramified along a $(2,2)$ divisor admits a natural map with image being a quartic Kummer surface in $\mathbb{P}^3$.

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