Differential invariants in the projective space

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The Monge invariant in the projective plane

P affine plane, with coordinates x, y
X family of curves in P of dimension n
A truncated arc (germ) of curve at a point (x, y) = a Taylor series (stopped at the order n)

$$(x + h, \Sigma_{0 \leq k \leq n} \frac{1}{k!} y^{(k)} h^k),$$

(or a thick point of length $n + 1$ on some smooth curve).
A differential equation of order n of X = a polynomial equation

$$f(x, y, y', \ldots, y^{(n)}) = 0$$

generically characteristic of the germs of order n of curves in X (X is the set of integral curves of $f = 0$.) The heuristics of elimination say that it exists.

The Monge invariant in the plane

A simple example is the differential equation of lines

$$y^{(2)} = 0.$$

A still easy example is

$$((y'')^{-2/3})''' = 0,$$

the 5-th order differential equation of conics, its rational form is

$$9(y'')^2 \times y^{(5)} - 45(y'')(y''')(y^{(4)}) + 40(y''')^3 = 0$$

(*Monge invariant.*)

These expressions can be seen as functions over the set of smooth arcs in the projective plane which are invariant under the projective group. We can imagine other such functions, for instance:

Given $d \in \mathbb{N}$, to find the differential equation of the PGL_3-orbit of the germ of order 8 (length 9) of a d-torsion point on a plane cubic

Definition of differential invariants

P=projective space of dimension n
$\mathcal{A}$=set of germs of smooth (algebroid) curves in P
Local charts on $\mathcal{P}$: choose coordinates $(X_0, \ldots, X_n)$.
(X_0): locus at infinity, (X_0, X_1): vertical directions
$\mathcal{A}_{01}$=set of germs not centered at infinity, with non-vertical tangent direction

Coordinates on $\mathcal{A}_{01}$:
A typical element of $\mathcal{A}_{01}$ takes the form

$$(1, a_{10} + t, \Sigma_{k\geq 0} a_{2k} t^k, \ldots, \Sigma_{k\geq 0} a_{nk} t^k),$$

its set of coordinates is the family $(a_{10}, (a_{ik})_{2\leq i\leq n, 0\leq k})$
Define similarly $\mathcal{A}_{ij}$ $0 \leq i, j \leq n, i \neq j$:
$\mathcal{A}$ becomes a smooth infinite-dimensional variety.

$Pic(\mathcal{A}) = \mathbb{Z}^2$, rule:
C curve in P, $\mathcal{C} \subset \mathcal{A}$ the corresponding curve of arcs

$$\mathcal{O}_{\mathcal{A}}(\delta, \varpi)|_{\mathcal{C}} = \omega_C^{\otimes\varpi}(\delta).$$

A *differential invariant* (on P) of degree δ and weight ϖ is a section of $\mathcal{O}_{\mathcal{A}}(\delta, \varpi)$ invariant under the natural action of $G = SL(A^*)$ on $\mathcal{A}$ ($A = H^0(\mathcal{O}_P(1))$).

Possible incarnations:
Any section of $\mathcal{O}_{\mathcal{A}}(\delta, \varpi)$ can be seen inside the coordinate ring of $\mathcal{A}_{01}$: a_{00} and $a_{00}a_{11} - a_{01}a_{10}$ are sections of $\mathcal{O}(1,0)$, resp $\mathcal{O}(2,1)$, nowhere vanishing on $\mathcal{A}_{01}$: they are identified to 1.
The differential invariants can also be seen inside the coordinate ring R of the grassmannian $\mathcal{G}_{n+1}$, locus of the $(n+1)$-planes in the vector space $\mathbb{C}[[t]]$.

Notation:

$$U_\lambda(a(t)) = det(a_{0,\lambda_0+n}, a_{1,\lambda_1+n-1}, \dots, a_{n,\lambda_n})$$

is the Plücker coordinate, of index λ, of a germ $a(t) \in A^* \otimes \mathbb{C}[[t]]$ (λ is a partition of length $\leq n+1$).
$U_0 = U$ is the differential invariant known as the *wronskian*. The other coordinates are no longer invariants.

Homogeneity questions.

The ring R is bigraded, the second graduation is given by the action of $\mathbb{C}^*$ on $\mathbb{C}[[t]]$ by the substitution of variable $t \mapsto at$, $a \in \mathbb{C}^*$; the bidegree of U_λ is $(1, |\lambda|)$.

Let I be an invariant of degree δ and weight ϖ. One has:

$\delta = (n+1)d$ for an integer d, the *reduced degree* of I;

$\varpi = \binom{n+1}{2}d + p$ for an integer p, the *reduced weight* of I;

the incarnation of I in R lies in the bihomogeneous component $R_{d,p}$.

A slice.

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The *slice method* is known to be helpful in computing rings of invariants:

Definition

Let X be a variety, G be a group acting on X, Y be a locally closed subvariety of X, H a subgroup of G stabilising Y. Then (Y, H) is a slice of (X, G) if the general G-orbit in X meets Y along a H-orbit.

Theorem

(Halphen, 1880) Consider the variety $\mathcal{A}$ of smooth arcs in P, acted on by the projective group PGL_{n+1}, and its open chart set $\mathcal{A}_{01}$. In $\mathcal{A}_{01}$ consider the closed set $\mathcal{S}$ whose typical element is

$$(1, t, t^2 + O(t^{n+3}), \ldots, t^{n-2} + O(t^{n+3}),$$

$$t^{n-1} + O(t^{n+4}), t^n + t^{n+3} + O(t^{n+4})).$$

Then $(\mathcal{S}, \mathbb{Z}_3)$ is a slice for $(G, \mathcal{A})$.

The union of translates of $\mathcal{S}$ under elements of G is the complement of an invariant hypersurface in $\mathcal{A}$. An easily guessed component of this hypersurface is the wronskian $U = 0$; there is exactly one other component whose equation is the extension, to the dimension n, of the *Monge invariant*, the differential equation of conics in the case $n = 2$. In the Plücker coordinates it is

$$\mu: = U^2(U_3 + 2U_{21} - 2U_{111}) - 3UU_1(U_{11} + U_2) + 2U_1^3 \in R_{3,3}.$$

This invariant has a reasonable geometric interpretation in the case $n = 3$, as the wronskian for the Plücker embedding of the developable (of the given arc). I don't know any such interpretation for higher n.

The invariant a_7 and the differential equation of plane cuspidal cubics.

Corollary

The quotient of the ring of differential invariants by the ideal $(U-1,\mu-1)$ *is the polynomial ring*

$$\mathbb{C}[(a_{ij})_{i\in[2,n-2],j\geq n+3},(a_{kl})_{k=n-1,n,l\geq n+4}].$$

In the case of the plane, $n=2$, the definition of the slice $\mathcal{S}$ is to be slightly modified: here $\mathcal{A}_{01}$ is the set of Taylor series

$$(1, a+t, \Sigma_{k\in\mathbb{N}} a_k t^k)$$

and the slice $\mathcal{S}\subset\mathcal{A}_{01}$ has the typical element $(1,t,t^2+t^5+O(t^7))$.

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In suitable affine coordinates the equation of the cuspidal cubic is $y=x^3$. More generally we may consider the curves whose equation in suitable affine coordinates are $y=x^\lambda$, or, equivalently, the trajectories of linear 3×3-differential systems

$$\frac{dX}{dt} = AX,$$

where A is a (constant) 3×3-matrix.

The G-orbit of a trajectory is a seven-dimensional family of analytic curves, whose differential equation is the quotient, by some monomial $U^\alpha \mu^\beta$, of a polynomial in (U, μ, Δ) where Δ is a differential invariant of order 7 representing the coordinate a_7 on $\mathcal{S}$.

The condition $\Delta = 0$ for an arc is its possible reduction to the form

$$(1, t, t^2 + t^5 + O(t^8)).$$

in suitable coordinates of P. An example is $y = x^2 + \frac{1}{x}$ at infinity (a branch of the double point of the *Descartes trifolium*). Finding the corresponding invariant differential equation is an easy exercise of elimination (in contrast to the cuspidal cubic!): the result is (in the coordinate ring of $\mathcal{A}_{01}$)

$$\Delta = \begin{vmatrix} a_3 & a_4 & a_5 & a_6 & a_7 \\ a_2 & a_3 & a_4 & a_5 & a_6 \\ -a_2^2 & 0 & a_3^2 & 2a_3a_4 & 2a_3a_5 + a_4^2 \\ 0 & a_2^2 & 2a_2a_3 & 2a_2a_4 + a_3^2 & 2a_2a_5 + 2a_3a_4 \\ 0 & 0 & a_2^2 & 3a_2a_3 & 3a_2a_4 + 3a_3^2 \end{vmatrix}.$$

Therefore $\Delta \in R_{8,8}$ and the rational function $I : = \frac{\Delta^3}{\mu^8}$ on $\mathcal{A}$ is an *absolute invariant*: it remains constant on each PGL_3-orbit.
In particular it is constant along the curve $y = x^\lambda$, which is invariant under a 1-parameter subgroup. The constant is found (by the binomial theorem) as

$$\frac{3^3.5^2}{2^4.7^3}.\frac{(1-\lambda+\lambda^2)^3}{((\lambda+1)(\lambda-2)(2\lambda-1))^2}$$

for I. Setting $\lambda = 3$, we find

$$(\frac{\Delta}{3})^3 - (\frac{\mu}{2})^8 = 0$$

for the differential equation of cuspidal cubics. This equation is divisible by U^4 in the ring of differential invariants.

Notice that the invariants μ and Δ can be viewed as the equations of sets of thick division points on cubics in the plane. Namely:

$\mu = 0$ is the equation of the set of points m on a variable smooth cubic C such that $\mathcal{O}_C[6m] = \mathcal{O}_C(2)$ (points of order 6), thickened to the length 6 on C;

$\Delta = 0$ is the equation of the set of points m on C such that $\mathcal{O}_C[9m] = \mathcal{O}_C(3)$ (points of order 9) thickened to the length 8 on C.