# Degenerations of Hilbert schemes of points

Martin G. Gulbrandsen[1]
joint w/Lars Halle and Klaus Hulek

[1]University of Stavanger

MCPG 2016

# Synopsis

- Hyperkähler manifolds as higher dimensional K3's: How do they degenerate?

# Synopsis

- Hyperkähler manifolds as higher dimensional K3's: How do they degenerate?
- Basic example: Semi-stable (symplectic?) degenerations of $\mathrm{Hilb}^n(K3)$

University of Stavanger

# Synopsis

- Hyperkähler manifolds as higher dimensional K3's: How do they degenerate?
- Basic example: Semi-stable (symplectic?) degenerations of $\mathrm{Hilb}^n(K3)$
- Li–Wu construction: $X \to C$ semi-stable degeneration without triple points $\rightsquigarrow$ improved version of $\mathrm{Hilb}^P(X/C)$

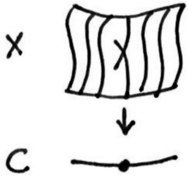
University of Stavanger

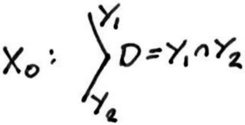
# Synopsis

- Hyperkähler manifolds as higher dimensional K3's: How do they degenerate?
- Basic example: Semi-stable (symplectic?) degenerations of $\mathrm{Hilb}^n(K3)$
- Li–Wu construction: $X \to C$ semi-stable degeneration without triple points $\rightsquigarrow$ improved version of $\mathrm{Hilb}^P(X/C)$
- Our aim:
  - apply Li–Wu to Hilbert scheme of points
  - in particular for K3 degenerations
  - make accessible by rephrasing as a GIT construction

# Example

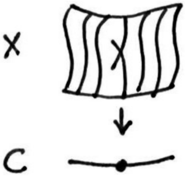
The curve degeneration $t = x_1 x_2$:

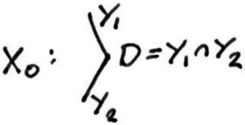


# Example

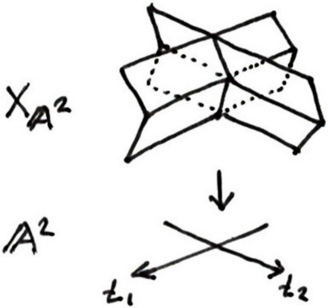
The curve degeneration $t = x_1 x_2$:



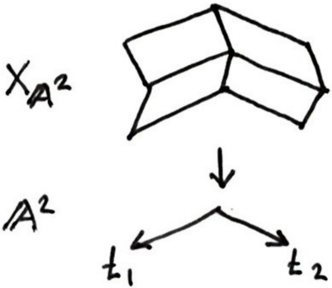

Impose singularityfobia:
describe the points in $X$ without touching the nonsmooth point, but rather as orbits in a smooth family.

# Example




- Pull back $X$ along product $\mathbb{A}^2 \to \mathbb{A}^1$: $t_1 t_2 = x_1 x_2$.

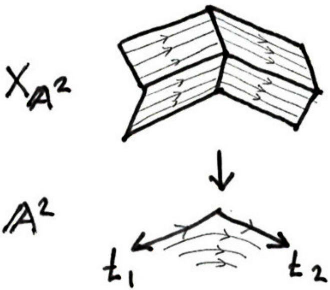
# Example




- Pull back $X$ along product $\mathbb{A}^2 \to \mathbb{A}^1$: $t_1 t_2 = x_1 x_2$.

University of Stavanger

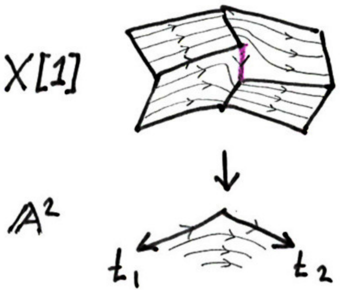
# Example




- Pull back $X$ along product $\mathbb{A}^2 \to \mathbb{A}^1$: $t_1 t_2 = x_1 x_2$.
  With $\mathbb{G}_m$-action $(t_1, t_2) \mapsto (\lambda t_1, \lambda^{-1} t_2)$.

University of Stavanger

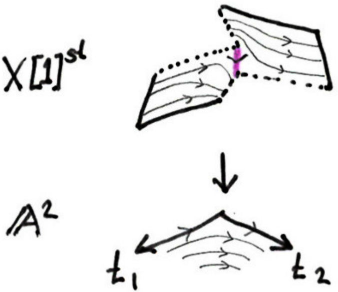
# Example




- Pull back $X$ along product $\mathbb{A}^2 \to \mathbb{A}^1$: $t_1 t_2 = x_1 x_2$.
  With $\mathbb{G}_m$-action $(t_1, t_2) \mapsto (\lambda t_1, \lambda^{-1} t_2)$.
- Choose a small resolution $X[1]$, e.g. blow up Weil divisor $x_1 = t_1 = 0$.

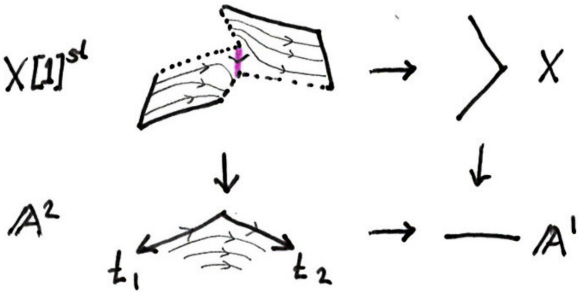
University of Stavanger

# Example




- Pull back $X$ along product $\mathbb{A}^2 \to \mathbb{A}^1$: $t_1 t_2 = x_1 x_2$.
  With $\mathbb{G}_m$-action $(t_1, t_2) \mapsto (\lambda t_1, \lambda^{-1} t_2)$.
- Choose a small resolution $X[1]$, e.g. blow up Weil divisor $x_1 = t_1 = 0$.
- Delete "$Y_1$ over the $t_1$-axis" and "$Y_2$ over the $t_2$-axis".

# Example




- Pull back $X$ along product $\mathbb{A}^2 \to \mathbb{A}^1$: $t_1 t_2 = x_1 x_2$.
  With $\mathbb{G}_m$-action $(t_1, t_2) \mapsto (\lambda t_1, \lambda^{-1} t_2)$.
- Choose a small resolution $X[1]$, e.g. blow up Weil divisor $x_1 = t_1 = 0$.
- Delete "$Y_1$ over the $t_1$-axis" and "$Y_2$ over the $t_2$-axis".
- $X$ is the $\mathbb{G}_m$-orbit space of the smooth variety $X[1]^{\mathrm{st}} \subset X[1]$.

University of Stavanger

# The Li–Wu stack

- Input:
  - semi-stable degeneration $X/C$ without triple points ($C = \mathbb{A}^1$)
  - orientation of the dual graph $\Gamma(X_0)$

University of Stavanger

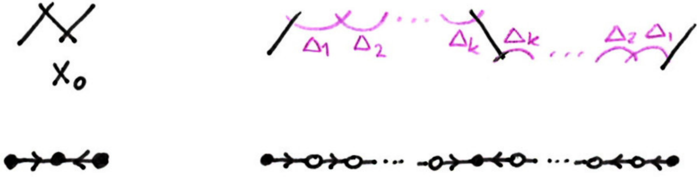
# The Li–Wu stack

- Input:
  - semi-stable degeneration $X/C$ without triple points ($C = \mathbb{A}^1$)
  - orientation of the dual graph $\Gamma(X_0)$
- Output: Deligne–Mumford stack $\mathfrak{I}^n_{X/C}$ such that
  - It is proper over $C$
  - It agrees with $\mathrm{Hilb}^n(X/C)$ over $C \setminus \{0\}$

University of Stavanger

# The Li–Wu stack: degenerate fibre

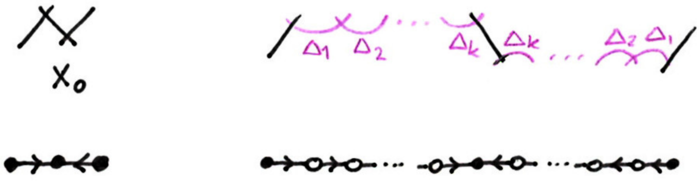
As a set: length $n$ subscheme in an *expansion* of $X_0$:




modulo $\mathbb{G}_m^k$ (scale each $\Delta_i$).

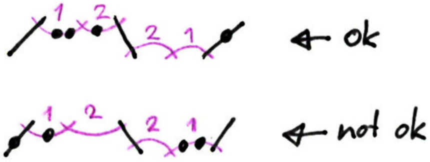
# The Li–Wu stack: degenerate fibre

As a set: length $n$ subscheme in an *expansion* of $X_0$:




modulo $\mathbb{G}_m^k$ (scale each $\Delta_i$). Subject to a "stability condition":

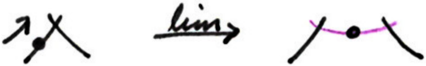
- No singularities as supporting points
- No empty $\Delta_i$




University of Stavanger

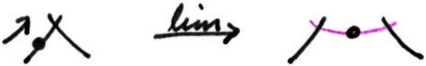
# The Li–Wu stack: families

Basic limit mechanism: avoid singularities by growing $\Delta$'s:



# The Li–Wu stack: families

Basic limit mechanism: avoid singularities by growing $\Delta$'s:




Families of expansions not entirely transparent: modelled on an explicitly constructed family $X[k]$ over $C[k] = C \times_{\mathbb{A}^1} \mathbb{A}^{k+1}$.

University of Stavanger

# The GIT construction

Assume: $X \to C$ is projective and $\Gamma(X_0)$ has no odd cycles. Choose bipartite orientation.

# The GIT construction

Assume: $X \to C$ is projective and $\Gamma(X_0)$ has no odd cycles. Choose bipartite orientation.

University of Stavanger

# The GIT construction

Assume: $X \to C$ is projective and $\Gamma(X_0)$ has no odd cycles. Choose bipartite orientation.

# The GIT construction

Assume: $X \to C$ is projective and $\Gamma(X_0)$ has no odd cycles. Choose bipartite orientation.

**Theorem (GHH)**

*$X[n] \to C[n]$ is projective and for a suitable $\mathbb{G}_m^n$-linearization on $H = \mathrm{Hilb}^n(X[n]/C[n])$:*

# The GIT construction

Assume: $X \to C$ is projective and $\Gamma(X_0)$ has no odd cycles. Choose bipartite orientation.

**Theorem (GHH)**

*$X[n] \to C[n]$ is projective and for a suitable $\mathbb{G}_m^n$-linearization on $H = \mathrm{Hilb}^n(X[n]/C[n])$:*

- *Stability and semi-stability coincides*
- *Stability means no singularities and no empty $\Delta_i$, but refined: zeros in the base $C[n]$ dictate distribution of points among the components.*

# The GIT construction

Assume: $X \to C$ is projective and $\Gamma(X_0)$ has no odd cycles. Choose bipartite orientation.

## Theorem (GHH)

*$X[n] \to C[n]$ is projective and for a suitable $\mathbb{G}_m^n$-linearization on $H = \mathrm{Hilb}^n(X[n]/C[n])$:*

- *Stability and semi-stability coincides*
- *Stability means no singularities and no empty $\Delta_i$, but refined: zeros in the base $C[n]$ dictate distribution of points among the components.*
- *The GIT quotient $H^{\mathrm{st}}/\mathbb{G}_m^n$ is the coarse moduli space of $\mathfrak{I}^n_{X/C}$.*
- *The Li–Wu stack $\mathfrak{I}^n_{X/C}$ is the global quotient $[H^{\mathrm{st}}/\mathbb{G}_m^n]$.*

# Properties of $\mathfrak{I}^n_{X/C}$

Work in progress! For $X \to C$ a degeneration of curves or surfaces, with bipartite orientation:

- $\mathfrak{I}^n_{X/C} \to C$ is *semi-stable*, in the stacky sense

University of Stavanger

# Properties of $\mathfrak{I}^n_{X/C}$

Work in progress! For $X \to C$ a degeneration of curves or surfaces, with bipartite orientation:

- $\mathfrak{I}^n_{X/C} \to C$ is *semi-stable*, in the stacky sense
- K3 degenerations: for $X \to C$ a Kulikov model of type II, $\mathfrak{I}^n_{X/C} \to C$ is (stacky) *good* in the sense of Nagai.

University of Stavanger

# Properties of $\mathfrak{I}^n_{X/C}$

Work in progress! For $X \to C$ a degeneration of curves or surfaces, with bipartite orientation:

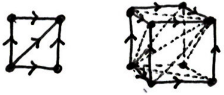
- $\mathfrak{I}^n_{X/C} \to C$ is *semi-stable*, in the stacky sense
- K3 degenerations: for $X \to C$ a Kulikov model of type II, $\mathfrak{I}^n_{X/C} \to C$ is (stacky) *good* in the sense of Nagai.
- The dual $\Delta$-complex of the degenerate fibre is the $n$'th symmetric power of the dual oriented graph of $X_0$
  - In particular: the dual graph $\Gamma$ linear: its symmetric product $S^n(\Gamma)$ is a subdivision of an $n$-simplex.

University of Stavanger

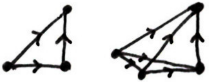
# The dual Δ-complex: 2 components, $n$ points

$\Gamma$ with orientation:

$\Gamma^n$ for $n = 2, 3, \ldots$:



$S^n(\Gamma)$ for $n = 2, 3, \ldots$:



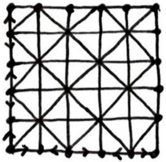
An $n$-simplex — simplest possible type for Nagai's good degenerations of hyperkähler varieties.

University of Stavanger

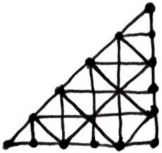
# The dual Δ-complex: $n$ components, 2 points

$\Gamma$ with orientation:

$\Gamma^2$:



$S^2(\Gamma)$:



Same combinatorics as Nagai's hand made examples.

University of Stavanger

# The dual Δ-complex: 3 components, 3 points:

Γ with orientation:

$S^3(\Gamma)$: