On birational transformations of $\mathbb{P}_3$ of degree 3 and 4

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Definition

$$\mathrm{Bir}_d(\mathbb{P}_3) = \left\{ \begin{array}{ll} \phi\colon \mathbb{P}_3 \dashrightarrow \mathbb{P}_3, & \text{birational linear system of degree d} \\ & \text{with base locus of codim} > 1 \end{array} \right\}$$

Definition

$$\mathrm{Bir}_d(\mathbb{P}_3) = \left\{ \phi : \mathbb{P}_3 \dashrightarrow \mathbb{P}_3, \quad \begin{array}{l} \text{birational linear system of degree d} \\ \text{with base locus of codim} > 1 \end{array} \right\}$$

Definition

$$\mathrm{Bir}_{d_1,d_2}(\mathbb{P}_3) = \{\phi \in \mathrm{Bir}_{d_1}(\mathbb{P}_3), \phi^{-1} \in \mathrm{Bir}_{d_2}(\mathbb{P}_3)\}$$

$\phi \in \mathrm{Bir}_d(\mathbb{P}_3)$, H hyperplane of $\mathbb{P}_3$ then

$\phi^{-1}(H)$ is a rational surface of degree d

$\mathrm{Bir}_d(\mathbb{P}_3)$		$\mathrm{Bir}_d(\mathbb{P}_3)$
$d \leq 3$	$\neq$	$d > 3$

so

$\mathrm{Bir}_{4,4}(\mathbb{P}_3)$?

(book 1927) Hudson's Table of $\mathrm{Bir}_3(\mathbb{P}_3)$

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TABLE VI

Cubic Space Transformations

Number	Degrees	D.p. of contact	Binode	D.p.'s	Pt. of osculation	Pt. of contact	Ordinary pts.	F-curves (1) denotes a branch touching a fixed biplane	Remarks	References
1	3—2	·	·	·	·	·	·	l^3, l_1, l_2, l_3	3 generators meet double line	
2	3—3	·	·	·	·	·	·	ω_6 (genus 3)		p. 287
3		·	·	1	·	·	·	$\omega_6 \equiv O^3$ (genus 3)		p. 294
4		1	·	·	·	·	·	$\omega_6 \equiv O^4$ (rational)		85
5		·	·	·	·	·	2	l^3, l_1, l_2	2 generators meet double line	85
6	3—4	·	·	·	·	·	1	ω_5 (genus 1)		p. 291
7		·	·	1	·	·	1	$\omega_5 \equiv O_1^2$ (genus 1)		p. 170
8		·	1	·	·	·	1	$\omega_5 \equiv O_1^3$ (2) (rational)		23, 97
9		1	·	·	·	·	1	$\omega_3 \equiv O_1^2, l_1 \equiv O_1, l_2 \equiv O_1$		
10		1	1	·	·	·	1	$\omega_2 \equiv O_1O_2, l \equiv O_1O_2$ (osculation)	(φ) touch plane along l	23
11		·	·	·	·	·	4	l^3, l_1	generator meets double line	
12	3—5	·	·	·	·	·	2	ω_3 (rational), l		p. 291
13		·	·	1	·	·	2	$\omega_4 \equiv O_1$ (genus 1)		p. 170
14		·	·	1	·	·	2	$\omega_3 \equiv O_1$ (rational), $l \equiv O_1$		85
15		·	1	·	·	·	2	$\omega_4 \equiv O_1^2$ (2)		
16		·	1	·	·	·	2	$\omega_2 \equiv O_1 (1), l_1 \equiv O_1 (1), l_2 \equiv O_1$		
17		1	·	·	·	·	2	$\omega_3 \equiv O_1^2, l_1 \equiv O_1$		
18		1	·	·	·	·	2	$l \equiv O_1$ (contact), $l_1 \equiv O_1, l_2 \equiv O_1$	(φ) touch quadric	
19		·	·	2	·	·	2	$\omega_3 \equiv O_1O_2$ (rational), $l \equiv O_1O_2$		
20		·	1	1	·	·	2	$\omega_2 \equiv O_1 (1) O_2, l_1 \equiv O_1O_2, l_2 \equiv O_1 (1)$		85
21		1	·	1	·	·	2	$l \equiv O_1O_2$ (contact), $l_1 \equiv O_1, l_2 \equiv O_1$	(φ) touch plane	
22		1	1	·	·	·	2	$l \equiv O_1O_2$ (1) (osculation), $l_1 \equiv O_1$	(φ) touch plane	
23		·	·	·	·	1	·	ω_4 (rational)		
24		·	·	1	·	1	·	$\omega_4 \equiv O_1^2$		49, p. 170
25		·	1	·	·	1	·	$\omega_3 \equiv O_1^2 (1), l \equiv O_1 (1)$		
26		1	·	·	·	1	·	$l_1 \equiv O_1, l_2 \equiv O_1, l_3 \equiv O_1, l_4 \equiv O_1$		
27		·	·	·	·	·	6	l^3		
28	3—6	·	·	·	·	·	3	l (contact), l_1		p. 291
29		·	·	1	·	·	3	ω_2 (plane, genus 1)		p. 170
30		·	·	1	·	·	3	$\omega_2, l \equiv O_1$		85, 258
31		·	·	1	·	·	3	$l \equiv O_1$ (contact), l_1		
32		·	·	1	·	·	3	$l \equiv O_1$ (osculation)	(φ) touch quadric	385
33		·	1	·	·	·	3	$\omega_2 \equiv O_1 (1), l \equiv O_1$		
34		1	·	·	·	·	3	$\omega_2 \equiv O_1^2$		
35		1	·	·	·	·	3	$l \equiv O_1$ (contact), $l_1 \equiv O_1$	(φ) touch quadric	
36		·	·	2	·	·	3	$\omega_2 \equiv O_1, l \equiv O_1O_2$		
37		·	1	1	·	·	3	$\omega_2 \equiv O_1 (1) O_2, l \equiv O_1O_2$		85
38		·	1	1	·	·	3	$l \equiv O_1O_2, l_1 \equiv O_1 (1), l_2 \equiv O_1 (1)$		
39		1	·	1	·	·	3	$l \equiv O_1O_2$ (contact), $l_1 \equiv O_1$	(φ) touch plane	
40		1	1	·	·	·	3	$l \equiv O_1O_2$ (1) (osculation)	(φ) touch plane	
41		·	·	·	·	1	1	l_1, l_2, l_3		
42		·	·	1	·	1	1	$\omega_2 \equiv O_1$ (rational)		41, p. 170
43		·	·	1	·	1	1	$l_1 \equiv O_1, l_2 \equiv O_1, l_3$		
44		·	1	·	·	1	1	$\omega_2 \equiv O_1^2 (1)$		
45		·	1	·	·	1	1	$\omega_2 \equiv O_1 (1), l \equiv O_1 (1)$		

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TABLE VI (*continued*)

Cubic Space Transformations

Number	Degrees	D.p. of contact	Binode	D.p.'s	Pt. of osculation	Pt. of contact	Ordinary pts.	F-curves (1) denotes a branch touching a fixed biplane	Remarks	References
5	3—6	·	1	·	·	1	1	$l \equiv O_1 (1)$ (contact), $l_1 \equiv O_1$	(φ) touch quadric	
7		1	·	·	·	1	1	$l_1 \equiv O_1, l_2 \equiv O_1, l_3 \equiv O_1$		
3		·	·	2	·	1	1	$l \equiv O_1O_2, l_1 \equiv O_1, l_2 \equiv O_2$		
		·	1	1	·	1	1	$l \equiv O_1 (1) O_2$ (contact), $l_2 \equiv O_1$	(φ) touch plane; O_2 on fixed plane at O_1	
		·	·	3	·	1	1	$l_1 \equiv O_2O_3, l_2 \equiv O_3O_1, l_3 \equiv O_1O_2$		
		·	·	·	1	·	·	ω_3 (rational)		85
		·	·	1	1	·	·	$\omega_3 \equiv O_1^2$		p. 170
	3—7	1	·	·	·	·	4	$l \equiv O_1$ (contact)	(φ) touch quadric	
		·	·	2	·	·	4	$l \equiv O_1O_2, l_1$		85
		·	1	1	·	·	4	$l \equiv O_1O_2, l_1 \equiv O_1 (1)$		
		1	·	1	·	·	4	$l \equiv O_1O_2$ (contact)	(φ) touch plane	
		·	·	1	·	1	2	ω_2		85
		·	1	·	·	1	2	$\omega_2 \equiv O_1 (1)$		
		1	·	·	·	1	2	$l_1 \equiv O_1, l_2 \equiv O_1$		
		·	·	1	·	2	·	$l \equiv O_1, l_1$		85
		·	1	·	·	2	·	$l_1 \equiv O_1 (1), l_2 \equiv O_1$		
		·	1	·	·	2	·	$l \equiv O_1 (1)$ (contact)	(φ) touch quadric	
		·	·	2	·	2	·	$l \equiv O_1O_2, l_1 \equiv O_1$		
		·	1	1	·	2	·	$l \equiv O_1 (1) O_2$ (contact)	(φ) touch plane; O_2 on fixed plane at O_1	
		·	·	1	1	·	1	$\omega_2 \equiv O_1$		
		·	1	·	1	·	1	$l_1 \equiv O_1 (1), l_2 \equiv O_1 (1)$		85, 258
	3—8	·	1	1	·	·	5	$l \equiv O_1O_2$		
		1	·	·	·	1	3	$l \equiv O_1$		
		·	1	·	·	2	1	$l \equiv O_1$		
		·	·	1	1	·	2	l		
		·	1	·	1	·	2	$l \equiv O_1 (1)$		85
	3—9	1	·	·	·	1	4			
		·	1	·	·	3	·			396
		·	1	·	1	·	3			
		·	·	1	1	·	2		4-point contact at O_2	18, 85, 385

Hudson's Table in bidegree (3,5)

D.p. of contact	binode	D. p.'s	pt of osc.	pt of contact	ord. pts	F-curves
·	·	·	·	·	2	ω_3 (rational), l
·	·	1	·	·	2	$\omega_4 \equiv O_1$ (genus 1)
·	·	1	·	·	2	$\omega_3 \equiv O_1$ (rational), $l \equiv O_1$
·	1	·	·	·	2	$\omega_4 \equiv O_1^2(2)$
·	1	·	·	·	2	$\omega_2 \equiv O_1(1)$, $l_1 \equiv O_1(1)$, $l_2 \equiv O_1$
1	·	·	·	·	2	$\omega_3 \equiv O_1^2$, $l_1 \equiv O_1$
1	·	·	·	·	2	$l \equiv O_1$ (contact), $l_1 \equiv O_1$, $l_2 \equiv O_1$
·	·	2	·	·	2	$\omega_3 \equiv O_1 O_2$ (rational), $l \equiv O_1 O_2$
·	1	1	·	·	2	$\omega_2 \equiv O_1(1) O_2$, $l_1 \equiv O_1 O_2$, $l_2 \equiv O_1(1)$
1	·	1	·	·	2	$l \equiv O_1 O_2$ (contact), $l_1 \equiv O_1$, $l_2 \equiv O_1$
1	1	·	·	·	2	$l \equiv O_1 O_2(1)$ (osculation), $l_1 \equiv O_1$
·	·	·	·	1	·	ω_4 (rational)
·	·	1	·	1	·	$\omega_4 \equiv O_1^2$
·	1	·	·	1	·	$\omega_3 \equiv O_1^2(1)$, $l \equiv O_1(1)$
1	·	·	·	1	·	$l_1 \equiv O_1$, $l_2 \equiv O_1$, $l_3 \equiv O_1$, $l_4 \equiv O_1$
·	·	·	·	·	6	l^2

Rough ideas about $\mathrm{Bir}_3(\mathbb{P}_3)$

The situation in $\mathrm{Bir}_3(\mathbb{P}_3)$ is poor with non normal surfaces

Only one component of $\mathrm{Bir}_{3,d}(\mathbb{P}_3)$ for each $2 \leq d \leq 5$

But

The situation in $\mathrm{Bir}_3(\mathbb{P}_3)$ is very rich with normal surfaces

Determinantal cubo-cubic

- $0 \to \mathcal{O}_{\mathbb{P}_3}^{\oplus 3}(-1) \xrightarrow{M} \mathcal{O}_{\mathbb{P}_3}^{\oplus 4} \to \mathscr{I}_C(3) \to 0$, ($M$ generic)
- $\mathscr{I}_C$ ideal of a curve C of degree 6 and genus 3.

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- $\widetilde{\mathbb{P}_3}(C)$ is a complete intersection in $\mathbb{P}_3 \times \mathbb{P}_3$. ($\sim (1,1)^3$)

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- The linear system $|\mathscr{I}_C(3)| : \mathbb{P}_3 \dashrightarrow |\mathscr{I}_C(3)|^\vee$ is birational

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Definition (Determinantal cubo-cubic)

Let $\mathcal{D}_{3,3} \subset \mathrm{Bir}_{3,3}(\mathbb{P}_3)$ be the set of maps defined by the maximal minors of a linear map $\mathcal{O}_{\mathbb{P}_3}^{\oplus 3}(-1) \to \mathcal{O}_{\mathbb{P}_3}^{\oplus 4}$.

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Remark

$\overline{\mathcal{D}_{3,3}}$ is an irreducible component of $\mathrm{Bir}_{3,3}(\mathbb{P}_3)$.

Determinantal quarto-quartic

- Is there a similar construction with quartics ?
- $0 \to \mathcal{O}_{\mathbb{P}_3}^{\oplus 2}(-1) \oplus \mathcal{O}_{\mathbb{P}_3}(-2) \xrightarrow{G} \mathcal{O}_{\mathbb{P}_3}^{\oplus 4} \to \mathscr{I}(4) \to 0,$

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- Problem, if G is generic : $\mathbb{P}_3 \overset{2:1}{\dashrightarrow} \mathbb{P}_3$

Example (Déserti, −)

- Let $\mathscr{I}_\Gamma$ be the ideal of a general trigonal curve $\Gamma \subset \mathbb{P}_3$ of degree 8 and genus 5,
- $\mathscr{I}_\Delta$ be the ideal of Δ, the unique line 5-secant to Γ.
- $0 \to \mathcal{O}_{\mathbb{P}_3}^{\oplus 2}(-1) \oplus \mathcal{O}_{\mathbb{P}_3}(-2) \xrightarrow{G} \mathcal{O}_{\mathbb{P}_3}^{\oplus 4} \to \mathscr{I}_\Delta^2 \cap \mathscr{I}_\Gamma(4) \to 0$
- The linear system $|\mathscr{I}_\Delta^2 \cap \mathscr{I}_\Gamma(4)| : \ \mathbb{P}_3 \dashrightarrow |\mathscr{I}_\Delta^2 \cap \mathscr{I}_\Gamma(4)|^\vee$ is birational

Construction in $\widetilde{\mathbb{P}_3}(\Delta)$

- $\widetilde{\mathbb{P}_3}(\Delta)$ = blow up of $\mathbb{P}_3$ in a line Δ.
- $X \subset \widetilde{\mathbb{P}_3}(\Delta) \times \mathbb{P}_3$ a complete intersection $(1,0,1)\cdot(0,1,1)\cdot(1,1,1)$

  $\cap$
- $X \subset \mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}_3$ a complete intersection $(1,1,0)\cdot(1,0,1)\cdot(0,1,1)\cdot(1,1,1)$

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- $\mathbb{P}_3 \leftarrow X \rightarrow \mathbb{P}_3$ are birational.

Definition (Determinantal quarto-quartic)

$\mathrm{Bir}_{4,4}(\mathbb{P}_3) \supset \mathcal{D}_{4,4} = \{\phi \mid \exists \Delta, X \text{ such that } \phi : \mathbb{P}_3 \dashrightarrow X \rightarrow \mathbb{P}_3\}$

- as $X \subset \widetilde{\mathbb{P}_3}(\Delta) \times \mathbb{P}_3$ is a complete intersection $(1,0,1) \cdot (0,1,1) \cdot (1,1,1)$
- $p : X \to \widetilde{\mathbb{P}_3}(\Delta)$, $p_*(\mathcal{O}_X(2,2,1))$ gives :

$$0 \to \begin{array}{c} \mathcal{O}_{\widetilde{\mathbb{P}_3(\Delta)}}(-1,0) \\ \oplus \\ \mathcal{O}_{\widetilde{\mathbb{P}_3(\Delta)}}(0,-1) \\ \oplus \\ \mathcal{O}_{\widetilde{\mathbb{P}_3(\Delta)}}(-1,-1) \end{array} \xrightarrow{\widetilde{G}} \mathcal{O}_{\widetilde{\mathbb{P}_3(\Delta)}}^{\oplus 4} \to \mathscr{I}_Z(2,2) \to 0$$

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- A general trigonal curve of degree 8 and genus 5 with Δ as 5-secant line has the same resolution.

Explicit constructions over $\mathbb{P}_3$

- $L_2 = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_1}(1)), \quad A_4 = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_3}(1)), \quad A'_4 = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}'_3}(1)), \quad B\colon L_2 \xrightarrow{\sim} L_2^\vee$
- $X \subset \mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3$ a complete intersection $(1,1,0)\cdot(1,0,1)\cdot(0,1,1)\cdot(1,1,1)$

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$(1,1,0)$

$L_2^\vee \xrightarrow{N_0} A_4$

$(1,0,1)$

$L_2^\vee \xrightarrow{N_1} A'_4$

$(0,1,1)$

$A_4^\vee \xrightarrow{M} A'_4$

$(1,1,1)$

$$\begin{array}{rccc} T: & L_2^\vee & \to & \mathrm{Hom}(A_4^\vee, A'_4) \\ & \lambda & \mapsto & T_\lambda \end{array}$$

- $$N_1 \circ B \circ {}^t N_0, \ M, \ T_\lambda : \quad A_4^\vee \longrightarrow A'_4$$

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- $L_2 = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_1}(1)), \quad A_4 = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_3}(1)), \quad A'_4 = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}'_3}(1)), \quad B : L_2 \xrightarrow{\sim} L_2^\vee$

$(1,1,0)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
$L_2^\vee \xrightarrow{N_0} A_4$	$L_2^\vee \xrightarrow{N_1} A'_4$	$A_4^\vee \xrightarrow{M} A'_4$	$T : \begin{array}{ccc} L_2^\vee & \to & \mathrm{Hom}(A_4^\vee, A'_4) \\ \lambda & \mapsto & T_\lambda \end{array}$

- $$N_1 \circ B \circ {}^t N_0, \ M, \ T_\lambda : \quad A_4^\vee \longrightarrow A'_4$$

- $\forall z \in A_4^\vee, \quad g_1 = N_1 \circ B \circ {}^t N_0(z), \quad g_2 = M(z), \quad g_3 = T_{B \circ {}^t N_0(z)}(z)$

  $g_1 \wedge g_2 \wedge g_3 \in \bigwedge^3 A'_4 = A'^\vee_4, \quad (g_i)$ gives the 3 columns of G

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- $L_2 = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_1}(1)), \quad A_4 = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_3}(1)), \quad A'_4 = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}'_3}(1)), \quad B : L_2 \xrightarrow{\sim} L_2^\vee$

$(1,1,0)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
$L_2^\vee \xrightarrow{N_0} A_4$	$L_2^\vee \xrightarrow{N_1} A'_4$	$A_4^\vee \xrightarrow{M} A'_4$	$T : L_2^\vee \to \mathrm{Hom}(A_4^\vee, A'_4)$, $\lambda \mapsto T_\lambda$

- $$N_1 \circ B \circ {}^tN_0,\ M,\ T_\lambda : \quad A_4^\vee \longrightarrow A'_4$$
- $\forall z \in A_4^\vee, \quad g_1 = N_1 \circ B \circ {}^tN_0(z), \quad g_2 = M(z), \quad g_3 = T_{B\circ {}^tN_0(z)}(z)$

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- Minors of G and G' gives a birational map and its inverse.

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Let ϕ *be a general element of* $\mathcal{D}_{4,4}$, *and* Γ *be the associated trigonal curve of genus* 5 *embedded in* $\mathbb{P}_3$ *by* $\mathcal{O}_\Gamma(H)$, *then*

1. $\phi^{-1} \in \mathcal{D}_{4,4}$ *is also constructed from* Γ *but embedded in* $\mathbb{P}_3$ *by* $\mathcal{O}_\Gamma(H')$
2. $\mathcal{O}_\Gamma(H') = \omega_\Gamma^{\otimes 2}(-H)$

Contracted locus

$\phi : \mathbb{P}_3 \dashrightarrow \mathbb{P}_3, \phi \in \mathcal{D}_{4,4}$, then ϕ contracts

1. A ruled surface of degree 9 (triangles with vertices elements of the g_3^1 of Γ)

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1. A ruled surface of degree 9 (triangles with vertices elements of the g_3^1 of Γ)
2. The cubic surface containing Γ.

Classical examples

- $\mathcal{J}_{d,d} \subset \mathrm{Bir}_{d,d}(\mathbb{P}_3)$ (lift an automorphism of $\mathbb{P}_2$ with monoids)

$$(z_0 : z_1 : z_2 : z_3) \mapsto \left(z_0 : z_1 : z_2 : \frac{z_3 P_{d-1}(z_0, z_1, z_2) + P_d(z_0, z_1, z_2)}{z_3 Q_{d-2}(z_0, z_1, z_2) + Q_{d-1}(z_0, z_1, z_2)} \right)$$

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  - In another component of $\mathrm{Bir}_{4,4}(\mathbb{P}_3)$

With normal surfaces

Remind : $\mathcal{J}_{d,d} \subset \mathrm{Bir}_{d,d}(\mathbb{P}_3)$ (lift an automorphism of $\mathbb{P}_2$ with monoids)

Proposition

Any element of $\mathrm{Bir}_{4,4}(\mathbb{P}_3)$ *with normal quartics in its linear system is in* $\overline{\mathcal{J}_{4,4}}$

At least one more component in $\mathrm{Bir}_{4,4}(\mathbb{P}_3)$

- Let p be a point of $\mathbb{P}_3$ of ideal $\mathscr{I}_p = (z_0, z_1, z_2)$
- $Q_1 \in \mathrm{H}^0(\mathscr{I}_p(2))$, $Q_2 \in \mathrm{H}^0(\mathscr{I}_p^2(2))$, $f \in \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_3}(1))$
- A general point p_1 of $\mathbb{P}_3$

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$$\mathscr{I} = (f, Q_1)^2 \cap (Q_1, Q_2) \cap \mathscr{I}_p^2 \cap \mathscr{I}_{p_1}, \quad \text{then} \quad |\mathscr{I}(4)| : \mathbb{P}_3 \overset{1:1}{\dashrightarrow} |\mathscr{I}(4)|^\vee$$

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- These examples must be included in another component of $\mathrm{Bir}_{4,4}(\mathbb{P}_3)$
- So $\mathrm{Bir}_{4,4}(\mathbb{P}_3)$ has at least 5 components