Calabi–Yau manifolds in low codimension

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joint work with:
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Calabi-Yau – definition and questions

Definition

A Calabi-Yau threefold is a smooth complex projective threefold X satisfying:

1. $K_X = 0$
2. $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$

Main questions and conjectures:

- Classification
- Mirror symmetry conjectures
- Web conjecture

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Our main aims from the point of view of Calabi-Yau theory:

- Fill the need of new well described constructions that could help us see beyond the toric world.
- Understand special phenomena specific to low codimensional Calabi-Yau manifolds.

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4	Partial description: Reid	??

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Classify projectively normal Calabi-Yau threefolds in $\mathbb{P}^7$.

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With $s = c_1(E) + 2rt$ and $\mathrm{rk}(E) = 2r + 1$ we then have:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-2s - t) \to E^*(-s - t) \to E(-s) \to \mathcal{I}_X \to 0,$$

and $\omega_X = \mathcal{O}_X(t + 2s - n - 1)$.

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Theorem

If n is not divisible by 4 then a locally Gorenstein codimension 3 submanifold of $\mathbb{P}^{n+3}$ is Pfaffian if and only if it is sub-canonical.

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- Schreyer, Walter: If $X \subset \mathbb{P}^6$ is a Calabi-Yau threefold then $X = \mathrm{Pf}(\phi)$ for some $\phi : E^*(-1) \to E$, $a_i \in \mathbb{Z}$ and

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The Hartshorne-Rao module of X is the module $HR(X) = \oplus_{k=1}^{N} H^1(\mathcal{I}_X(k))$.

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In particular there is a finite number of families of Calabi-Yau threefolds in $\mathbb{P}^6$.

degree	Vector bundle
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We classify Calabi-Yau threefolds with simple Hartshorne-Rao module (trivial or with trivial multiplication); those contained in a quadric ; and Calabi-Yau threefolds of degree ≤ 14.

Three types in degree 17

The bundles for threefold of degree 17 are constructed in the following way. The map $\psi\colon 16\mathcal{O}_{\mathbb{P}^6} \to 3\mathcal{O}_{\mathbb{P}^6}(1)$ is given by a 16×3 matrix of linear forms i.e. its columns span a $\mathbb{P}^{15} \subset \mathbb{P}^{20} \supset \mathbb{P}^2 \times \mathbb{P}^6$.

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Proposition

The Calabi-Yau threefolds constructed in case 3 have Picard number ≥ 2.

Analogy with del Pezzo surfaces

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Using the analogy we can construct a canonical surface of degree 18 in $\mathbb{P}^5$

Structure theorem in codimension 4

Let X be a codimension 4 projectively Gorenstein variety then the ideal $\mathcal{I}_X$ admits a free resolution of the form:

$$0 \to P_4 \to P_3 \xrightarrow{M'} P_2 \xrightarrow{M} P_1 \xrightarrow{L} \mathcal{I}_X \to 0, \tag{1}$$

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with $M = (A, B)$ is a $(k+1) \times 2k$ matrix with polynomial entries made of two blocks A, B satisfying $A(B^t) + B(A^t) = 0$ and $M' = \begin{pmatrix} B^t \\ A^t \end{pmatrix}$.

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Theorem (Reid)

Conversely if $M = (A, B)$ is a matrix as above for which the rank $< k$ locus D_{k-1} satisfies codim $D_{k-1} \geq 4$ then there exists X projectively Gorenstein of codimension 4 with resolution (1).

Finiteness and degree bound

Proposition

The dimension of the space of quadrics in the ideal of a projectively normal Calabi–Yau threefold X in $\mathbb{P}^7$ of degree d is $20 - d$ i.e. $h^0(\mathcal{I}_X(2)) = 20 - d$. Moreover, the ideal of $X \subset \mathbb{P}^7$ is generated by quintics.

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The following degree bounds follow:

Proposition

Let X be a nonsingular projectively normal Calabi–Yau 3-fold in $\mathbb{P}^7$. The degree of X takes values between 14 and 20.

Known constructions and classification

Deg.	$h^{1,1}$	$h^{1,2}$	Description
14	2	86	$(2,4)$ type divisor in $\mathbb{P}^1 \times \mathbb{P}^3$
15	1	76	$G(2,5) \cap C_3 \cap H_1 \cap H_1'$
16	1	65	$X_{2,2,2,2}$
17	1	55	$Mv = \bigwedge^3 M = 0$, bilinked on $X_{2,2,2}$ to $\mathbb{P}^3$
17	2	58	2×2 minors of a matrix with degrees $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$
17	2	54	rolling factors, codim 2 in cubic scroll
18	1	46	bilinked on $X_{2,2,3} \subset \mathbb{P}^7$ to F_2
18	1	45	bilinked on $X_{2,2,3} \subset \mathbb{P}^7$ to F_1
19	2	36	bilinked on Pf_{13} to F_2
19	2	37	bilinked on Pf_{13} to F_1
20	2	34	3×3 minors of 4×4 matrix with linear forms

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18	1	46	bilinked on $X_{2,2,3} \subset \mathbb{P}^7$ to F_2
18	1	45	bilinked on $X_{2,2,3} \subset \mathbb{P}^7$ to F_1
19	2	36	bilinked on Pf_{13} to F_2
19	2	37	bilinked on Pf_{13} to F_1
20	2	34	3×3 minors of 4×4 matrix with linear forms

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- Its intersection with a linear subspace $\mathbb{P}^6$ is a del Pezzo surface $S_6 \subset \mathbb{P}^6$.
- Let $Y_{13} \subset \mathbb{P}^6$ be a general (singular) Calabi-Yau threefold of degree 13 containing S_6.

The unprojection

Such a Y_{13} and S_6 are defined as degeneracy loci of matrices:

$$Y_{13} : \begin{pmatrix} A & B & C & D \\ & z_{31} & z_{21} & z_{22} - z_{33} \\ & & z_{11} & z_{12} \\ & & & z_{13} \end{pmatrix} \text{ and } S_6 : \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix},$$

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We can know perform a **Kustin-Miller unprojection**

Proposition (Kustin-Miller)

There exists a Gorenstein variety $Z \subset \mathbb{P}^7$ singular in a point p such that the projection of Z from p is Y_{13} and the exceptional locus of the projection is S_6.

Smoothing

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$$\begin{tikzcd}
\mathbb{P}^2 \times \mathbb{P}^2 \arrow[r, hook] & \Theta_{13} \arrow[r, hook] & \mathbb{P}^8 \arrow[r, dashed] & \mathbb{P}^8 & \mathbb{P}^7 \arrow[l, hook'] & \Sigma_{2,3} \arrow[l, hook'] \\
S_6 \arrow[u, hook] \arrow[r, hook] & Y_{13} \arrow[u, hook] \arrow[r, hook] & \mathbb{P}^7 \arrow[u, hook] \arrow[r, dashed] & \mathbb{P}^8 \arrow[u, equal] & \mathbb{P}^7 \arrow[u, equal] \arrow[l, hook'] & \Sigma_{2,3} \cap Q_{2,2} \arrow[u, hook] \arrow[l, hook']
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The image $\Sigma_{2,3}$ of Θ_{13} is a complete $\mathbb{P}^6$ section of the secant variety of $\mathbb{P}^2 \times \mathbb{P}^3$.

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Thank you