Fano congruences of index 3 and alternating 3-forms

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Mediterranean Complex Projective Geometry in Carry-le-Rouet
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Outline

Joint work with Piero De Poi, Daniele Faenzi, and Kristian Ranestad

K algebraically closed field, $\mathrm{char}(K) = 0$

Definition

A congruence of lines in $\mathbb{P}^n$ is a family of lines of dimension $n-1$, i.e. a subvariety of dimension $n-1$ of the Grassmannian $\mathbb{G} = \mathbb{G}(1, n) = G(2, V)$, V vector space of dimension $n+1$.

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The order of a congruence $X \subset G(2, V)$ is the number of lines of X passing through a general point of $\mathbb{P}(V)$.

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The fundamental locus F of a congruence of lines X is the set of points of $\mathbb{P}^n$ contained in infinitely many lines of X.

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Natural class of examples

Linear congruences of lines $=$ proper linear sections of $\mathbb{G}$ with a linear subspace Λ of codimension $n-1$.

The order of a linear congruence is one.

For general Λ, the linear congruence $X = \mathbb{G} \cap \Lambda$ is smooth irreducible and a Fano variety of index 2,
for particular choices it can be singular or reducible.

Example

$n = 3$, $X =$ smooth quadric surface: lines meeting two skew lines
$n = 4$, $X = \mathbb{G}(1,4) \cap \mathbb{P}^6$: trisecant lines of a projected Veronese surface
$n = 5$, $X = \mathbb{G}(1,5) \cap \mathbb{P}^{10}$: $4-$ secant lines of the Palatini threefold

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Examples

Other examples of nonlinear congruences of order one:

- trisecant lines of a Bordiga surface in $\mathbb{P}^4$
- secant lines of a OADP (=one apparent double point) variety

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History

- **Congruences of lines in $\mathbb{P}^3$**
  Kummer: first classification results of congruences of order one and two.
  Schumacher, Bordiga, C. Segre, Fano, Semple, Roth.
  N. Goldstein: classification problem from the point of view of the focal locus.
  Z. Ran: surfaces of order one in Grassmannians
  Arrondo, Sols, Gross, Turrini, Bertolini, Verra...
- **Congruences of lines of order one in $\mathbb{P}^n, n > 3$**
  Castelnuovo, Palatini, Marletta and his student Sgroi; De Poi
- Applications of congruences of order one: problems of rationality, OADP varieties, degree of irrationality of hypersurfaces, systems of conservation laws...

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Congruence associated to a 3-form

We study congruences defined by 3-forms, a class of congruences that are irreducible components of some reducible linear congruences, and their residual.

Definition

Fix $\omega \in \bigwedge^3 V^*$ general 3-form

$\Lambda_\omega = \{[L] \in \mathbb{P}(\bigwedge^2 V) \mid \omega(L) = 0\} \subset \mathbb{P}(\bigwedge^2 V)$ is a linear subspace of codimension $n+1$ (if $n > 3$)

$X_\omega = \Lambda_\omega \cap \mathbb{G}$ is an improper intersection: a congruence

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$n = 3$: $\alpha-$plane: lines through a point F_ω

$n = 4$: quadric threefold in a $\mathbb{G}(1,3)$, F_ω is the $\mathbb{P}^3$

$n = 5$: $\mathbb{P}^2 \times \mathbb{P}^2$, the lines meeting two skew planes

$n = 6$: homogeneous G_2 variety, closed orbit of the $\mathbf{G}_2$ group in its adjoint representation, F_ω a smooth quadric (M. Kapustka – Ranestad)

$n = 7$: variety of reductions of Severi variety $\mathbb{P}^2 \times \mathbb{P}^2$, trisecant lines of a general projection of $\mathbb{P}^2 \times \mathbb{P}^2$ (Iliev – Manivel)

$n = 8$: a family of lines in the Coble cubic hypersurface (Gruson – Sam)

$n = 9$: variety of $4-$secant lines of a $6-$dimensional smooth non $2-$normal variety, at the border of Zak's conjectures on $k-$normality (Peskine, Han)

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- X_ω is irreducible smooth
- Its geometry depends on the parity of n:
  - n even: order 0, family of lines in a hypersurface of degree $\frac{n}{2} - 1$
  - n odd: order 1, $\frac{n-1}{2}$- secant lines of F_ω, of codimension 3
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The residual congruence

A subspace of codimension $n-1$ containing Λ_ω depends on two linear forms $x, y \in V^*$:
$\Lambda_\omega^{xy} = \{[L] \in \mathbb{P}(\bigwedge^2 V) \mid \omega(L) \wedge x \wedge y = 0\}$.

The residual congruence of X_ω $Y := Y_{\omega, x\wedge y}$ is defined by the Gorenstein liaison

$$I_Y = [I_Z : I_{X_\omega}]$$

where $Z = \mathbb{G} \cap \Lambda_\omega^{xy}$. So

$$\mathbb{G} \cap \Lambda_\omega^{xy} = X_\omega \cup Y_{\omega, x\wedge y}$$

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X_ω as degeneracy locus

$$0 \to \mathcal{U} \to V \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{Q} \to 0,$$

$\mathcal{U}$ the universal subbundle of rank 2
$\mathcal{Q}$ the quotient bundle of rank $n-1$

$H^0(\mathcal{Q}) = V$, $H^0(\mathcal{U}^*) = V^*$, $H^0(\mathcal{Q}^*(1)) \simeq \bigwedge^3 V^* \Longrightarrow$

- ω defines a map $\varphi_\omega : \mathcal{O}_{\mathbb{G}} \to \mathcal{Q}^*(1)$
- X_ω is the zero locus of a section of $\mathcal{Q}^*(1)$
- for general ω, X_ω is smooth of codimension $n-1$
- locally free resolution which gives $\omega_{X_\omega} \simeq \mathcal{O}_{X_\omega}(-3)$:

$$0 \to \mathcal{O}_{\mathbb{G}}(2-n) \to \bigwedge^{n-2}(\mathcal{Q}(-1)) \to \cdots \to \mathcal{Q}(-1) \xrightarrow{{}^t\varphi_\omega} \mathcal{I}_{X_\omega} \to 0.$$

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Degree and multidegree

The cohomology class of X_ω is computed by Porteous formula, we get degree and multidegree of X_ω.

The multidegree is $(d_\ell(n)) = (a_{(n-1-\ell,\ell)}), \ell = 0, \dots, n-1$.

$$(a_{i,j}) = \begin{array}{llllllllll}
1 \\
0 & 0 \\
1 & 1 & 1 \\
0 & 1 & 2 & 2 \\
1 & 2 & 4 & 6 & 6 \\
0 & 2 & 6 & 12 & 18 & 18 \\
1 & 3 & 9 & 21 & 39 & 57 & 57 \\
0 & 3 & 12 & 33 & 72 & 129 & 186 & 186 \\
1 & 4 & 16 & 49 & 121 & 250 & 436 & 622 & 622 \\
0 & 4 & 20 & 69 & 190 & 440 & 876 & 1498 & 2120 & 2120
\end{array}.$$

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$$(a_{i,j}) = \begin{matrix} 1 \\ \color{red}{0} & 0 \\ 1 & \color{red}{1} & 1 \\ 0 & 1 & \color{red}{2} & 2 \\ 1 & 2 & 4 & \color{red}{6} & 6 \\ 0 & 2 & 6 & 12 & \color{red}{18} & 18 \\ 1 & 3 & 9 & 21 & 39 & \color{red}{57} & 57 \\ 0 & 3 & 12 & 33 & 72 & 129 & \color{red}{186} & 186 \\ 1 & 4 & 16 & 49 & 121 & 250 & 436 & \color{red}{622} & 622 \\ 0 & 4 & 20 & 69 & 190 & 440 & 876 & 1498 & \color{red}{2120} & 2120 \end{matrix} .$$

F_ω as degeneracy locus

Remark

$\bigwedge^3 V^* \simeq H^0(\Omega^2_{\mathbb{P}(V)}(3) \subset H^0(\Omega^1 \otimes \Omega^1(3)) \simeq$
$\mathrm{Hom}(\Omega^1(1)^*, \Omega^1(2)) \simeq \mathrm{Hom}(\mathcal{T}(-1), \Omega^1(2))$

therefore ω *defines a bundle map* $\phi_\omega : \mathcal{T}(-1) \to \Omega^1(2)$.

We get a commutative diagram

$$\begin{array}{ccc} \mathcal{T}_{\mathbb{P}(V)}(-1) & \xrightarrow{\phi_\omega} & \Omega^1_{\mathbb{P}(V)}(2) \\ \uparrow & & \downarrow \\ V \otimes \mathcal{O}_{\mathbb{P}(V)} & \xrightarrow{M_\omega} & V^* \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \end{array}$$

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$$\begin{tikzcd}
\mathcal{T}_{\mathbb{P}(V)}(-1) \arrow[r, "\phi_\omega"] & \Omega^1_{\mathbb{P}(V)}(2) \arrow[d] \\
V \otimes \mathcal{O}_{\mathbb{P}(V)} \arrow[u] \arrow[r, "M_\omega"] & V^* \otimes \mathcal{O}_{\mathbb{P}(V)}(1)
\end{tikzcd}$$

M_ω is a $(n+1)\times(n+1)$ skew – symmetric matrix of linear forms, of rank $\leq n$, that can be explicitly written.

The degeneracy locus of M_ω is F_ω, so

- if n is even, F_ω is a hypersurface of degree $\frac{n}{2}-1$, smooth if $n\leq 6$;
- if n is odd, F_ω is a codimension 3 subvariety of degree $\frac{1}{4}\binom{n-1}{3}+1$, smooth if $n\leq 9$;
- the lines in X_ω are $(\frac{n-1}{2})-$secant F_ω (n odd).

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The bundle map $\phi_\omega : \mathcal{T}(-1) \to \Omega^1(2)$ can be completed to an exact sequence which defines the cokernel sheaf $\mathcal{C}_\omega$:

- n even:

$$0 \to \mathcal{T}_{\mathbb{P}(V)}(-1) \to \Omega^1_{\mathbb{P}(V)}(2) \to \mathcal{C}_\omega \to 0;$$

- $n = 2m+1$:

$$0 \to \mathcal{O}_{\mathbb{P}(V)}(1-m) \to \mathcal{T}_{\mathbb{P}(V)}(-1) \to \Omega^1_{\mathbb{P}(V)}(2) \to \mathcal{I}_{F_\omega/\mathbb{P}(V)}(m) \to 0,$$

  where $\mathcal{C}_\omega \simeq \mathcal{I}_{F_\omega/\mathbb{P}(V)}(m)$
  $\Longrightarrow$ F_ω is a Fano variety of index 3.

- $\mathbb{P}(\mathcal{C}_\omega) \simeq I_\omega$, the point-line incidence variety restricted to X_ω.

The bundle map $\phi_\omega : \mathcal{T}(-1) \to \Omega^1(2)$ can be completed to an exact sequence which defines the cokernel sheaf $\mathcal{C}_\omega$:

- n even:

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$M_r \subset \mathbb{P}(V)$: the degeneracy locus where ϕ_ω has rank at most r.

- When n is odd, the incidence variety I_ω is the blow-up of $\mathbb{P}(V)$ along F_ω.
- When n is even, the restriction of the incidence variety I_ω is a $\mathbb{P}^1$-bundle over the smooth locus $F_\omega \setminus M_{n-4}$, and a $\mathbb{P}^{2k-1}$-bundle over $M_{n-2k} \setminus M_{n-2k-2}$, for $k = 2, ..., (n-2)/2$.

Example

$\mathcal{C}_\omega \mid_{F_\omega}$ is
$n = 4$: null-correlation bundle
$n = 6$: stable rank 2 bundle, G_2 homogeneous: Cayley bundle

Hilbert scheme

Define the open dense subset $\mathcal{K} \subset \bigwedge^3 V^*$ by the condition: $\omega \in \mathcal{K}$ if and only if $\dim X_\omega = n - 1$.
We get a natural morphism:

$$\rho \colon \mathbb{P}(\mathcal{K}) \to \mathcal{H},$$

$\mathcal{H}$: components of the Hilbert scheme of $\mathbb{P}(\bigwedge^2 V)$ containing $[X_\omega]$.

Theorem

For $n \geq 5$, $\mathcal{H}$ is irreducible and smooth at any point $[X_\omega]$ where X_ω has expected dimension. Moreover:

(i) *for $n \geq 6$, ρ is a birational embedding to an open dense subset of $\mathcal{H}$, so* $\dim(\mathcal{H}) = \binom{n+1}{3} - 1$;

(ii) *for $n = 5$, ρ is dominant with rational curves as fibres, so* $\dim(\mathcal{H}) = \binom{n+1}{3} - 2$.

$$\mathcal{Q}_\omega = \{ \text{ quadrics containing } X_\omega \text{ and } \mathbb{G}\}.$$

Quadrics containing $\mathbb{G}$ are parametrized by $\bigwedge^4 V^*$:
$\eta \in \bigwedge^4 V^*$ defines $Q_\eta : \eta(L \wedge L) = 0$.

1. If η is totally decomposable, rank $Q_\eta = 6$;
2. If $\eta = \beta \wedge x \wedge x' \neq 0$, and $\beta_{x,x'}$ is the restriction of β to $\{x = x' = 0\}$, then rank $Q_\eta = 2\,\mathrm{rank}\,\beta_{x,x'} + 2 \leq 2n$.
3. If $\eta = \omega \wedge x$, where $x \neq 0$ and ω_x is the restriction of ω to $\{x = 0\}$, then rank $Q_\eta = 2\,\mathrm{rank}\,\omega_x \leq 2n$.

Theorem

- $I(\mathbb{G})_2 \cap I(\Lambda_\omega) = \mathcal{Q}_\omega = \{Q_{\omega\wedge x} \mid x \in V^*\}$ *(for general ω)*
- $\dim \mathcal{Q}_\omega = n + 1$

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Linear spaces in quadrics of $\mathcal{Q}_\omega$.

A general quadric $Q_{\omega\wedge x}$ in $\mathcal{Q}_\omega$ has rank $2n$, it contains two families of maximal isotropic spaces.

- $\Lambda_\omega = \langle X_\omega \rangle$ has codimension one in Λ^x_ω, space of one of the two families;
- $\operatorname{Sing} Q_{\omega\wedge x} = \Lambda_{\omega_x} = \langle X_{\omega_x} \rangle$, ω_x restriction of ω to the hyperplane $V_x = \{x = 0\}$.

Moreover

- $\Lambda^x_\omega \cap \mathbb{G} = X_\omega \cup X_{\omega_x}$;
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Geometry of Y_ω

Consider a general $Y = Y_{\omega,x\wedge y}$, residual congruence to X_ω

$$\mathbb{G} \cap \Lambda_\omega^{xy} = X_\omega \cup Y_{\omega,x\wedge y}$$

- $\Lambda_\omega \subset \Lambda_\omega^{x} \subset \Lambda_\omega^{xy}$
- $\langle Y \rangle = \Lambda_{\omega,x\wedge y} \subset \Lambda_\omega^{xy}$, maximal isotropic space in the second family in $Q_{\omega\wedge x}$;
- $Y = \Lambda_{\omega,x\wedge y} \cap \mathbb{G}$: improper intersection but codimension n;
- $X_\omega \cap Y$: lines of X meeting $\{x = y = 0\}$;
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$\Pi = \{x = y = 0\}$: for any hyperplane $ax + by = 0$ in the pencil with support Π, we have $X_{\omega_{[a:b]}}$.

Theorem

1. $$Y_{\omega,x\wedge y} = \cup_{[a:b]\in\mathbb{P}^1} X_{\omega_{[a:b]}}.$$
2. *The singular locus of* $Y_{\omega,x\wedge y}$ *is*
$$\bigcap X_{\omega_{[a:b]}} = \{L \in G(2,\Pi) \mid \omega_{x\wedge y}(L) = \omega_x(L) = \omega_y(L) = 0\},$$
where $\omega_{x\wedge y}$ *is the restriction of* ω *to* Π. *In particular the codimension of the singular locus of* Y *is* 4.
3. *For* $n \leq 4$, $K_Y \simeq X_{\omega_x} - 3H_Y$.

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Fundamental locus of Y

Theorem

$Y_{\omega,x\wedge y}$ residual congruence of X_ω with respect to $\Pi = \{x = y = 0\}$
G fundamental locus of Y

- *n odd: G is a hypersurface of degree $\frac{n-1}{2}$ containing Π and F_ω, the fundamental locus of X_ω;*
- *$n = 2m$ even: $G = \Pi \cup G_0$, $G_0 \subset F_\omega$,*
  $\deg G_0 = 2\binom{m+1}{3} + \binom{m+1}{2} + 2$; the lines of Y are $\frac{n-2}{2}-$ secant of G_0 meeting also Π;
- *G_0 is the residual component of $G_1 = \Pi \cap F_\omega$ in the zero locus of a global section of $\mathcal{C}_\omega\,|_{F_\omega}$.*

Examples

- $n = 3$ Y : lines of a plane $G = \langle F_\omega, \Pi \rangle$
- $n = 4$ $Y = \mathbb{P}^1 \times \mathbb{P}^2$, lines meeting Π and $G_0 = \mathbb{P}^1$
- $n = 5$ Y : fourfold of degree 8 in $\mathbb{P}^9$ with a singular point, 4-dimensional family of lines of a rank 4 quadric G
- $n = 6$ Y : variety of degree 24, secant lines of a smooth rational normal scroll of dimension 3, meeting also Π. Sing Y is a conic corresponding to $G_0 \cap \Pi = G_1 \cap F_{\omega_{x \wedge y}}$
- $n = 7$ Y: 6-dimensional family of lines of a cubic hypersurface
- $n = 8$ Y: trisecant lines of a 5-dimensional variety of degree 32 meeting also a $\mathbb{P}^6$.

Thank you