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A geometric characterization of flag manifolds

Gianluca Occhetta

with R. Muñoz, L.E. Solá Conde, K. Watanabe and J. Wiśniewski

Carry-le-Rouet, May 2016

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Fano bundles

Definition

A vector bundle \mathcal{E} on a smooth complex projective variety X is called a Fano bundle iff $\mathbb{P}_{X}(\mathcal{E})$ is a Fano manifold.

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Fano bundles

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If \mathcal{E} is a Fano bundle on X then X is a Fano manifold.

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Some classification results:

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Some classification results:

- $otin Fano bundles of rank 2 on <math>\mathbb{P}^m$ and \mathbb{Q}^m [Ancona, Peternell, Sols, Szurek, Wiśniewski]

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Some classification results:

- ☑ Fano bundles of rank 2 on del Pezzo threefolds [Szurek & Wiśniewski]

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Generalization: Classification of Fano bundles of rank 2 on (Fano) manifolds with $b_2=b_4=1$ (MOS, 2012).

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As a special case we have the classification of Fano manifolds of Picard number two (and $b_4=2$) with two \mathbb{P}^1 -bundle structures.

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As a special case we have the classification of Fano manifolds of Picard number two (and $b_4=2$) with two \mathbb{P}^1 -bundle structures.

Later the assumption on b_4 was removed by Watanabe (2013).

Finally the assumption " \mathbb{P}^1 -bundle" was replaced by "smooth \mathbb{P}^1 -fibration" (MOSWa 2014).

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Varieties with two \mathbb{P}^1 -fibrations

Theorem 1

Varieties with two \mathbb{P}^1 -fibrations

Theorem 1

A Fano manifold with Picard number 2 whose elementary contractions are \mathbb{P}^1 -fibrations is isomorphic to one of the following

• $\mathbb{P}_{\mathbb{P}^1}(\mathbb{O} \oplus \mathbb{O})$

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Theorem 1

- $\mathbb{P}_{\mathbb{P}^1}(\mathbb{O} \oplus \mathbb{O})$
- $\bullet \ \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$

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- $\mathbb{P}_{\mathbb{P}^1}(\mathfrak{O} \oplus \mathfrak{O})$
- $\mathbb{P}_{\mathbb{P}^2}(\mathsf{T}_{\mathbb{P}^2})$
- $\mathbb{P}_{\mathbb{P}^3}(\mathcal{N}) = \mathbb{P}_{\mathbb{O}^3}(\mathcal{S})$ \mathcal{N} Null-correlation , \mathcal{S} Spinor

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- $\mathbb{P}_{\mathbb{Q}^5}(\mathcal{C}) = \mathbb{P}_{K(G_2)}(\mathcal{Q})$ \mathcal{C} Cayley, \mathcal{Q} universal quotient.

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Remark

All the varieties appearing in the list are rational homogeneous

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Problem

Try to classify Fano manifolds whose elementary contractions are \mathbb{P}^1 -bundles - or just smooth \mathbb{P}^1 -fibrations.

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Problem

Try to classify Fano manifolds whose elementary contractions are \mathbb{P}^1 -bundles - or just smooth \mathbb{P}^1 -fibrations.

- The vector bundle approach seems difficult to apply to this more general situation.
- Is it possible to prove directly that these varieties are rational homogeneous?

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Rational homogeneous manifolds

Definition

A Borel subgroup B of a semisimple Lie group G is a maximal closed, connected solvable algebraic subgroup. A subgroup $P \supseteq B$ is called a parabolic subgroup.

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Rational homogeneous manifolds

Definition

A Borel subgroup B of a semisimple Lie group G is a maximal closed, connected solvable algebraic subgroup. A subgroup $P \supseteq B$ is called a parabolic subgroup.

For example, if $G = \operatorname{SL}_{n+1}$, then the subgroup of invertible upper triangular matrices is a Borel subgroup, while the parabolic subgroups correspond to $\emptyset \neq I \subseteq \{1, \ldots, n\}$.

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If $I = \{\alpha_1, \dots, \alpha_k\}$ and $\alpha_{k+1} := n+1$, then P(I) is the subgroup

$$\begin{pmatrix} B_1 & * & * & * \\ 0 & B_2 & * & * \\ 0 & 0 & \dots & * \\ 0 & 0 & 0 & B_{k+1} \end{pmatrix}$$

where the B_i 's are square matrices of order $a_i - a_{i-1}$.

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Definition

A rational homogeneous manifold is the quotient of a semisimple Lie group G by a parabolic subgroup P.

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Rational homogeneous manifolds

Definition

A rational homogeneous manifold is the quotient of a semisimple Lie group G by a parabolic subgroup P.

For example, if $G = SL_{n+1}$, setting

- $\{e_1, \dots, e_{n+1}\}$ standard basis of \mathbb{C}^{n+1} ;
- $I = \{a_1, ..., a_k\} \subseteq \{1, ..., n\};$
- $W_{\alpha_i} = \langle \mathbf{e}_1, \dots, \mathbf{e}_{\alpha_i} \rangle$

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- $W_{a_i} = \langle \mathbf{e}_1, \dots, \mathbf{e}_{a_i} \rangle$
- P(I) is the stabilizer w.r.t. the SL_{n+1} -action of the flag

$$W_{\alpha_1} \subset W_{\alpha_2} \subset \cdots \subset W_{\alpha_k}$$
.

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- P(I) is the stabilizer w.r.t. the SL_{n+1} -action of the flag

$$W_{\alpha_1} \subset W_{\alpha_2} \subset \cdots \subset W_{\alpha_k}$$
.

So G/P(I) is the variety $\mathbb{F}^n(a_1,\ldots,a_k)$ of flags of subspaces of dimensions a_1,\ldots,a_k of \mathbb{C}^{n+1} .

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We can denote the variety $\mathbb{F}^n(\alpha_1,\dots,\alpha_k)$ by a marked diagram.

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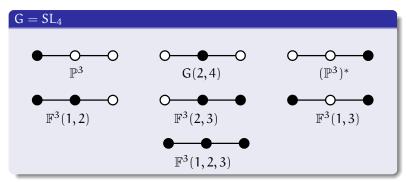
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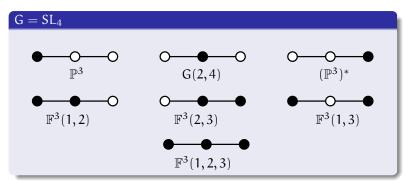
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Rational homogeneous manifolds

We can denote the variety $\mathbb{F}^n(a_1,\ldots,a_k)$ by a marked diagram.



The diagram used is the Dynkin diagram of the Lie algebra \$l_4:



Dynkin diagrams

Dynkin diagrams

- G semisimple Lie group,
- g associated Lie algebra,
- n rank of g.

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Dynkin diagrams

- G semisimple Lie group,
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- n rank of g.

Parabolic subgroups again correspond to $\emptyset \neq I \subseteq \{1, \ldots, n\}$, and the variety G/P(I) is denoted by marking the Dynkin diagram of $\mathfrak g$ along the nodes corresponding to I.

$$G/P(I) \leftrightarrow (\mathfrak{D}, \mathfrak{I})$$

Dynkin diagrams

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$$G/P(I) \leftrightarrow (\mathfrak{D}, \mathfrak{I})$$

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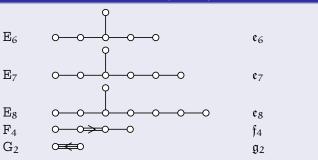
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Dynkin diagrams of the exceptional (simple) Lie algebras



Dynkin diagrams of rank two semisimple Lie algebras









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X Rational Homogeneous given by $(\mathcal{D}, \mathcal{I})$.

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Cone and contractions

X Rational Homogeneous given by $(\mathfrak{D}, \mathfrak{I})$.

• X is a Fano manifold of Picard number $\rho_X = \#I$;

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- X is a Fano manifold of Picard number $\rho_X = \#I$;
- The cone NE(X) is simplicial, and its faces correspond to proper subsets $J\subsetneq I$;

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Cone and contractions

- X is a Fano manifold of Picard number $\rho_X = \#I$;
- The cone NE(X) is simplicial, and its faces correspond to proper subsets $J \subsetneq I$;
- Every contraction $\pi: X \to Y$ is of fiber type and smooth.

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Cone and contractions

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- Y is RH with marked Dynkin diagram $(\mathcal{D}, \mathcal{J})$,

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Cone and contractions

- X is a Fano manifold of Picard number $\rho_X = \#I$;
- The cone NE(X) is simplicial, and its faces correspond to proper subsets J ⊊ I;
- Every contraction $\pi: X \to Y$ is of fiber type and smooth.
- Y is RH with marked Dynkin diagram $(\mathfrak{D}, \mathfrak{J})$,
- Every fiber is RH with marked Dynkin diagram $(\mathcal{D} \setminus \mathcal{J}, \mathcal{I} \setminus \mathcal{J})$.

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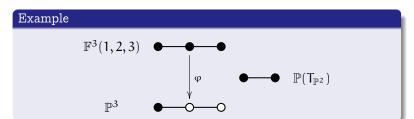
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Complete flag manifolds

Definition

A complete flag manifold is a RH manifold with a diagram in which all the nodes are marked. i.e. a quotient G/B by a Borel subgroup.

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 Every RH manifold is dominated by a complete flag manifold. RH manifolds
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Complete flag manifolds

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A complete flag manifold is a RH manifold with a diagram in which all the nodes are marked. i.e. a quotient G/B by a Borel subgroup.

- Every RH manifold is dominated by a complete flag manifold.
- $p_i:G/B\to G/P^i$ contractions corresponding to the unmarking of one node are \mathbb{P}^1 -fibrations.

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Complete flag manifolds

Definition

A complete flag manifold is a RH manifold with a diagram in which all the nodes are marked. i.e. a quotient G/B by a Borel subgroup.

- Every RH manifold is dominated by a complete flag manifold.
- $p_i:G/B\to G/P^i$ contractions corresponding to the unmarking of one node are \mathbb{P}^1 -fibrations.
- If Γ_i is a fiber of p_i , and K_i the relative canonical, the intersection matrix $[-K_i \cdot \Gamma_j]$ is the Cartan matrix of the Lie algebra \mathfrak{g} .

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Flag manifolds

Fano bundles and flag manifolds

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Fano bundles and flag manifolds

Theorem 1

A Fano manifold with Picard number 2 whose elementary contractions are \mathbb{P}^1 -fibrations is isomorphic to one of the following

$$\mathbb{P}_{\mathbb{P}^1}(\mathbb{O} \oplus \mathbb{O}) \qquad \mathbb{P}_{\mathbb{P}^2}(\mathsf{T}_{\mathbb{P}^2})$$

$$\mathbb{P}_{\mathbb{P}^2}(\mathsf{I}_{\mathbb{P}^2})$$

$$\mathbb{P}_{\mathbb{P}^3}(\mathcal{N})$$

$$\mathbb{P}_{\mathbb{Q}^5}(\mathcal{C})$$









Flag manifolds

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$$\mathbb{P}_{\mathbb{Q}^5}(\mathcal{C})$$











Theorem 1'

A Fano manifold with Picard number 2 whose elementary contractions are \mathbb{P}^1 -fibrations is a complete flag manifold.

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Theorem 2

A Fano manifold X whose elementary contractions are \mathbb{P}^1 -fibrations is a complete flag manifold G/B, for some semisimple group G.

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Theorem 2

A Fano manifold X whose elementary contractions are \mathbb{P}^1 fibrations is a complete flag manifold G/B, for some semisimple group G.

Strategy:

- 1) Find a homogeneous model G/B for X.
- 2) Prove that $X \simeq G/B$.

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Idea of proof

Part 1) - Finding a model

The flag manifold G/B is determined by the Lie algebra \mathfrak{g} , and the Lie algebra \mathfrak{g} is determined by any one of the following data:

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Idea of proof

Part 1) - Finding a model

The flag manifold G/B is determined by the Lie algebra \mathfrak{g} , and the Lie algebra \mathfrak{g} is determined by any one of the following data:

- its associated root system $\Phi \subset \mathbb{R}^n$;
- its Cartan matrix $A = [a_{ij}] \in M_n(\mathbb{Z})$;
- its Dynkin diagram D.
- its Weyl group W.

Part 1) - Finding a model

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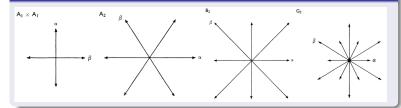
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Root systems of rank two semisimple Lie algebras



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Lemma

$$\begin{split} H^i(M,D) &\cong & H^{i-1}(M,D+lK) & \text{ if } l < 0 \\ H^i(M,D) &\cong & \{0\} & \text{ if } l = 0 \\ H^i(M,D) &\cong & H^{i+1}(M,D+lK) & \text{ if } l > 0 \end{split}$$

$$\label{eq:linear_equation} \mbox{In particular} \ \ \, X(M,D) = -X(M,D + (D \cdot \Gamma + 1)K) \ \ \, \mbox{for any } D.$$

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Lemma

$$H^{i}(M, D) \cong H^{i-1}(M, D + lK)$$
 if $l < 0$
 $H^{i}(M, D) \cong \{0\}$ if $l = 0$
 $H^{i}(M, D) \cong H^{i+1}(M, D + lK)$ if $l > 0$

In particular
$$X(M,D) = -X(M,D+(D\cdot\Gamma+1)K)$$
 for any D .

	l	 -3	-2	-1	0	1	2	3	
	H ⁰	 0	0	0	0	1	2	3	
Ì	H^1	 3	2	1	0	0	0	0	

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 $H^{i}(M, D) \cong H^{i+1}(M, D + lK)$ if $l > 0$

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Lemma

$$H^{i}(M, D) \cong H^{i-1}(M, D + lK)$$
 if $l < 0$
 $H^{i}(M, D) \cong \{0\}$ if $l = 0$
 $H^{i}(M, D) \cong H^{i+1}(M, D + lK)$ if $l > 0$

In particular
$$X(M,D) = -X(M,D+(D\cdot\Gamma+1)K)$$
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- X Fano manifold with Picard number n.
- $\pi_i: X \to X_i$ elementary contration (\mathbb{P}^1 -fibration).
- K_i relative canonical, Γ_i fiber of π_i .
- $X_X : Pic(X) \to \mathbb{Z}$ such that $X_X(L) = X(X, L)$.

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Given L_1, \ldots, L_n basis of Pic(X),

$$X_X(m_1,\ldots,m_n)=X(X,m_1L_1+\cdots+m_nL_n)$$

is a numerical polynomial of degree dim X; we can thus extend it to a function $\chi_X: N_1(X) \to \mathbb{R}$.

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We define also $T:N_1(X)\to N_1(X)$ and $X_T:N_1(X)\to \mathbb{R}$ as

$$T(D) := D + K_X/2 \hspace{1cm} X_T = X_X \circ T$$

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$$\mathsf{T}(\mathsf{D}) := \mathsf{D} + \mathsf{K}_\mathsf{X}/2 \qquad \qquad \mathsf{X}_\mathsf{T} = \mathsf{X}_\mathsf{X} \circ \mathsf{T}$$

Note that $T(D) \cdot \Gamma_i = D \cdot \Gamma_i - 1$.

Reflections

Define • hyperplanes $M_i := \{D \mid D \cdot \Gamma_i = 0\}$

• linear involutions $r_i: N^1(X) \to N^1(X)$ as

$$r_{\mathfrak{i}}(D) = D + (D \cdot \Gamma_{\mathfrak{i}}) K_{\mathfrak{i}}$$

Then

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1 r_i fixes pointwise the hyperplane M_i .

$$r_i(K_i) = -K_i$$

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$$\ \, \textbf{4} \ \, \textbf{X}^T|_{\textbf{M}_{\mathfrak{i}}} \equiv \textbf{0}$$

Proof of 3). Pick D in the lattice $-K_X/2 + Pic(X)$; then

$$\begin{split} X^T(D) &= X(T(D)) = -X(T(D) + (T(D) \cdot \Gamma_i + 1)K_i) \\ &= -X(T(D) + (D \cdot \Gamma_i)K_i) = -X(T(r_i(D)) \\ &= X^T(r_i(D)) \end{split}$$

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Let $W \subset Gl(N^1(X))$ be the group generated by the r_i 's.

$$\chi^T(D) = \pm \chi^T(w(D)), \qquad \forall D \in N_1(X), \quad \forall w \in W.$$

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Theorem

The group W is finite and

$$\Phi := \{ w(-K_i) \mid w \in W, \ i = 1, ..., n \} \subset N^1(X),$$

is a root system, whose Weyl group is W and whose Cartan matrix is the intersection matrix $[-K_i \cdot \Gamma_i]$.

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Idea of proof.

 X_X^T vanishes on the hyperplanes $w(M_i)$; therefore the number of these hyperplanes is bounded by the dimension of X.

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Then one proves that the isotropy subgroup of M_i is finite by considering the induced action on $N_1(X)$, and writing the elements of W is a suitable basis.

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Then one proves that the isotropy subgroup of M_i is finite by considering the induced action on $N_1(X)$, and writing the elements of W is a suitable basis.

By the finiteness there is a W-invariant scalar product (,) on $N^1(X)$. In particular the r_i 's are euclidean reflections.

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Since (,) is W-invariant, $(K_j,K_i)=(r_i(K_j),-K_i)$ which gives

$$\langle K_j, K_i \rangle := 2 \frac{(K_j, K_i)}{(K_i, K_i)} = -K_j \cdot \Gamma_i,$$

so the Cartan matrix of Φ is the intersection matrix $[-K_j \cdot \Gamma_i]$.

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Idea of Proof

Part 2) - Proving the isomorphism

X Fano manifold of Picard number n whose elementary contractions are \mathbb{P}^1 -fibrations. With $\ell=(l_1,\ldots,l_t),$ list of indices in $\{1,\ldots,n\}$ we can associate

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$$X_\ell := \pi_{l_t}^{-1}(\pi_{l_t}(\ldots(\pi_{l_2}^{-1}(\pi_{l_2}(\pi_{l_1}^{-1}(\pi_{l_1}(x)))))))$$

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3 A smooth t-dimensional variety Z_{ℓ} , with a morphism $f_{\ell}: Z_{\ell} \to X_{\ell}$, which is a tower of \mathbb{P}^1 -bundles.

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Bott-Samelson varieties

Set $\ell[1] = (l_1, \dots, l_{t-1})$. Then the Bott-Samelson variety Z_ℓ associated with ℓ , is constructed in the following way:

• If $\ell = \emptyset$ set $Z_{\ell} := \{x\}$ and let $f_{\ell} : \{x\} \to X$ be the inclusion.

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- If $\ell = \emptyset$ set $Z_{\ell} := \{x\}$ and let $f_{\ell} : \{x\} \to X$ be the inclusion.
- Z_{ℓ} is built inductively on $Z_{\ell[1]}$:

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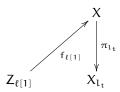
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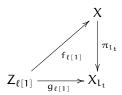
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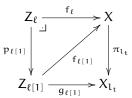
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Definition

If there is no factorization of $w(\ell)$ in less than $\#(\ell)$ simple reflections, then $w(\ell)$ and ℓ are called reduced.

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- 2 The morphism $f_\ell: Z_\ell \to X_\ell$ is birational

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One can prove that ℓ is reduced if and only if

- **1** The dimension of X_{ℓ} is $\#(\ell)$
- 2 The morphism $f_{\ell}: Z_{\ell} \to X_{\ell}$ is birational

In W there exists a unique longest element w_0 , such that if ℓ_0 is a reduced list such that $w(\ell_0) = w_0$ then $\#(\ell_0) = \dim X$.

In particular $f_{\ell}: Z_{\ell_0} \to X$ is surjective and birational.

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- $\overline{X} \simeq G/B$ homogeneus model of X,
- ℓ_0 list such that $w(\ell_0) = w_0$,
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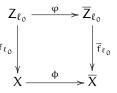
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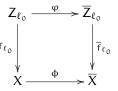
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The idea is to show inductively that Z_{ℓ_0} depends only on the list and on the intersection matrix, and that f_{ℓ_0} , \bar{f}_{ℓ_0} are contractions of the same face of the cone of curves.

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Theorem 2'

X smooth projective variety of Picard number $\mathfrak n$, such that there exist $\mathfrak n$ extremal rays, whose associated elementary contractions $\pi_i:X\to X_i$ are smooth $\mathbb P^1$ -fibrations. Then X is isomorphic to a flag manifold G/B, for some semisimple group G.