A geometric characterization of flag manifolds

Gianluca Occhetta

with R. Muñoz, L.E. Solá Conde, K. Watanabe and J. Wiśniewski

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Fano bundles

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- ☑ Fano bundles of rank 2 on del Pezzo threefolds [Szurek & Wiśniewski]

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Finally the assumption "$\mathbb{P}^1$-bundle" was replaced by "smooth $\mathbb{P}^1$-fibration" (MOSWa 2014).

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Remark

All the varieties appearing in the list are rational homogeneous

A generalization

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- The vector bundle approach seems difficult to apply to this more general situation.
- Is it possible to prove directly that these varieties are rational homogeneous?

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If $I = \{a_1, \dots, a_k\}$ and $a_{k+1} := n + 1$, then $P(I)$ is the subgroup

$$\begin{pmatrix} B_1 & * & * & * \\ 0 & B_2 & * & * \\ 0 & 0 & \dots & * \\ 0 & 0 & 0 & B_{k+1} \end{pmatrix}$$

where the B_j's are square matrices of order $a_j - a_{j-1}$.

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$$W_{a_1} \subset W_{a_2} \subset \cdots \subset W_{a_k}.$$

So $G/P(I)$ is the variety $\mathbb{F}^n(a_1, \dots, a_k)$ of flags of subspaces of dimensions $a_1, \dots, a_k$ of $\mathbb{C}^{n+1}$.

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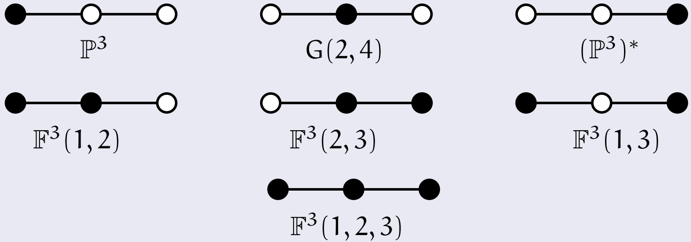

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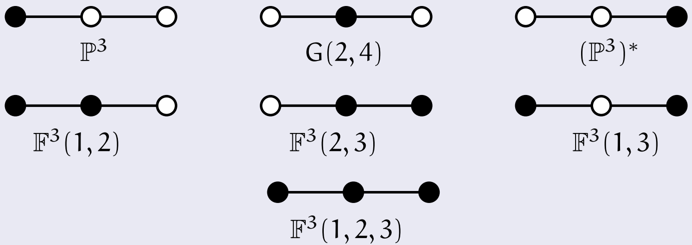
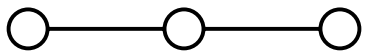


The diagram used is the Dynkin diagram of the Lie algebra $\mathfrak{sl}_4$:



Dynkin diagrams

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Parabolic subgroups again correspond to $\emptyset \neq I \subseteq \{1, \dots, n\}$, and the variety $G/P(I)$ is denoted by marking the Dynkin diagram of $\mathfrak{g}$ along the nodes corresponding to I.

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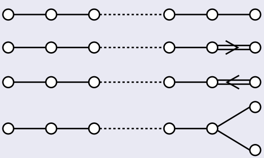
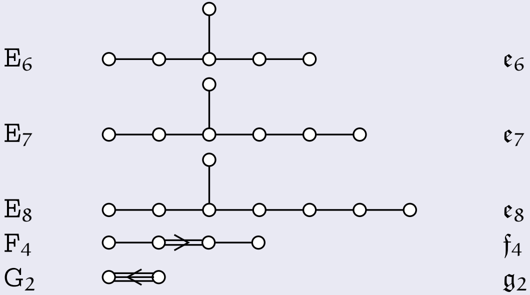
Dynkin diagrams of the classical (simple) Lie algebras



A_n	$\mathfrak{sl}_{n+1}$
B_n	$\mathfrak{so}_{2n+1}$
C_n	$\mathfrak{sp}_{2n}$
D_n	$\mathfrak{so}_{2n}$

Dynkin diagrams

Dynkin diagrams of the exceptional (simple) Lie algebras



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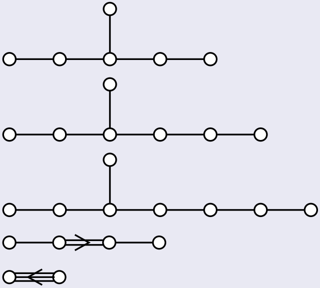
E_6 $\mathfrak{e}_6$

E_7 $\mathfrak{e}_7$

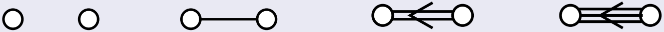
E_8 $\mathfrak{e}_8$

F_4 $\mathfrak{f}_4$

G_2 $\mathfrak{g}_2$



Dynkin diagrams of rank two semisimple Lie algebras



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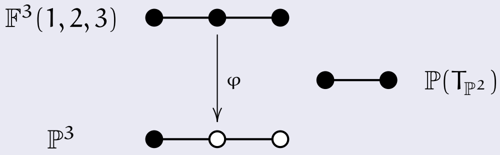
Example



Complete flag manifolds

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- If Γ_i is a fiber of p_i, and K_i the relative canonical, the intersection matrix $[-K_i \cdot \Gamma_j]$ is the Cartan matrix of the Lie algebra $\mathfrak{g}$.

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Main result

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Strategy:

1) Find a homogeneous model G/B for X.
2) Prove that $X \simeq G/B$.

Idea of proof

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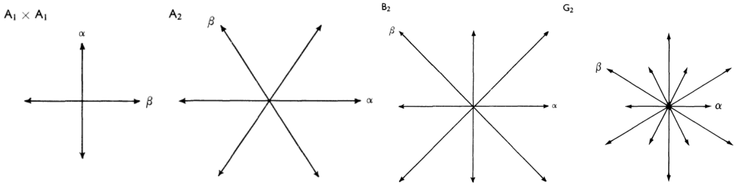
Root systems of rank two semisimple Lie algebras



Relative duality

Lemma

$\pi : M \to Y$ smooth $\mathbb{P}^1$-fibration. Γ fiber, K relative canonical. Let D be a divisor on M and set $l := D \cdot \Gamma + 1$. Then, $\forall i \in \mathbb{Z}$

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\begin{aligned}
H^i(M, D) &\cong H^{i-1}(M, D + lK) && \textit{if } l < 0 \\
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In particular $\chi(M, D) = -\chi(M, D + (D \cdot \Gamma + 1)K)$ *for any* D.

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Given $L_1, \dots, L_n$ basis of $\mathrm{Pic}(X)$,

$$\chi_X(m_1, \dots, m_n) = \chi(X, m_1 L_1 + \cdots + m_n L_n)$$

is a numerical polynomial of degree $\dim X$; we can thus extend it to a function $\chi_X : N_1(X) \to \mathbb{R}$.

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Note that $T(D) \cdot \Gamma_i = D \cdot \Gamma_i - 1$.

Reflections

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Proof of 3). Pick D in the lattice $-K_X/2 + \mathrm{Pic}(X)$; then

$$\begin{aligned} \chi^T(D) &= \chi(T(D)) = -\chi(T(D) + (T(D)\cdot\Gamma_i + 1)K_i) \\ &= -\chi(T(D) + (D\cdot\Gamma_i)K_i) = -\chi(T(r_i(D))) \\ &= \chi^T(r_i(D)) \end{aligned}$$

Homogeneous model

Let $W \subset \mathrm{Gl}(N^1(X))$ be the group generated by the r_i's.

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Theorem

The group W is finite and

$$\Phi := \{w(-K_i) \mid w \in W, \ \ i = 1, \dots, n\} \subset N^1(X),$$

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Idea of proof.

χ_X^T vanishes on the hyperplanes $w(M_i)$; therefore the number of these hyperplanes is bounded by the dimension of X.

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Since $(\ ,\ )$ is W-invariant, $(K_j, K_i) = (r_i(K_j), -K_i)$ which gives

$$\langle K_j, K_i \rangle := 2\frac{(K_j, K_i)}{(K_i, K_i)} = -K_j \cdot \Gamma_i,$$

so the Cartan matrix of Φ is the intersection matrix $[-K_j \cdot \Gamma_i]$. ■

Idea of Proof

Part 2) - Proving the isomorphism

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3. A smooth t-dimensional variety Z_ℓ, with a morphism $f_\ell : Z_\ell \to X_\ell$, which is a tower of $\mathbb{P}^1$-bundles.

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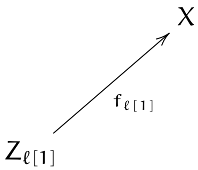

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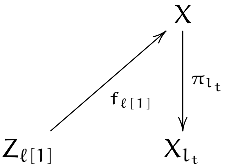

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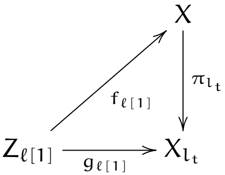

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$$\begin{array}{ccc} Z_\ell & \xrightarrow{f_\ell} & X \\ \downarrow p_{\ell[1]} & \nearrow f_{\ell[1]} & \downarrow \pi_{l_t} \\ Z_{\ell[1]} & \xrightarrow[g_{\ell[1]}]{} & X_{l_t} \end{array}$$

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In W there exists a unique longest element w_0, such that if ℓ_0 is a reduced list such that $w(\ell_0) = w_0$ then $\#(\ell_0) = \dim X$.

In particular $f_\ell : Z_{\ell_0} \to X$ is surjective and birational.

Conclusion

- $\overline{X} \simeq G/B$ homogeneus model of X,
- ℓ_0 list such that $w(\ell_0) = w_0$,
- $Z_{\ell_0}, \overline{Z}_{\ell_0}$ Bott-Samelson varieties of X and $\overline{X}$.

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$$\begin{array}{ccc} Z_{\ell_0} & \xrightarrow{\varphi} & \overline{Z}_{\ell_0} \\ {\scriptstyle f_{\ell_0}}\downarrow & & \downarrow{\scriptstyle \overline{f}_{\ell_0}} \\ X & \xrightarrow{\Phi} & \overline{X} \end{array}$$

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The idea is to show inductively that Z_{ℓ_0} depends only on the list and on the intersection matrix, and that $f_{\ell_0}, \overline{f}_{\ell_0}$ are contractions of the same face of the cone of curves.

A generalization

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Theorem 2'

X smooth projective variety of Picard number n, such that there exist n extremal rays, whose associated elementary contractions $\pi_i : X \to X_i$ are smooth $\mathbb{P}^1$-fibrations. Then X is isomorphic to a flag manifold G/B, for some semisimple group G.