Superficial Fibres of Generic Projections

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Outline

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Vision

Vision problem: Given a geometric object X, described by finitely many functions $x_1, ..., x_N$ (real or complex-valued) i.e. the mapping $(x_1, ..., x_N)$ is one to one,
then choose randomly a subspace W of given dimension in the space V spanned by the x_i.
How much information is lost by passing from V to W?
In other words, for a basis $y_1, ..., y_m$ of W, how far is the mapping $(y_1, ..., y_m)$ from being one to one ?
Analogue of 'seeing' a 3-dimensional object on a 2-dimensional retina.

Set-up

Algebro-geometric version: $X \subset \mathbb{P}^N =: P$ smooth subvariety of dim n, codim c
$\Lambda = \mathbb{P}^\lambda \subset P$ disjoint from X, $\lambda \geq 0$.
$G_\lambda = \mathbb{G}(\lambda + 1, P)$ parameter space for $(\lambda + 1)$-spaces in P.
$\Sigma_\Lambda = \{L \in G_\lambda : \Lambda \subset L\} \simeq \mathbb{P}^{N-\lambda-1}$
$\pi_\Lambda : X \to \mathbb{P}^{N-\lambda-1}$ projection, with image $= \bar{X}$,
then we ask: how singular is $\bar{X}$?

Multisecants and expectations

Singularities of $\bar{X}$ have to do with multisecant spaces of X:
Secant locus:
$Sec_k(X)_G = \{L \in G : length(X \cap L) \geq k\}$
It has expected codimension $k(c - \lambda - 1)$ in G.
More generally, for a partition $(k.) = (k_1 \geq ... \geq k_r > 0)$, we say $X \cap L$ is of type $(k.)$ if it has the form $\sum k_i p_i, p_i \in X$, strictly so if the p_i are distinct.
Then we have the contact locus

$$Sec_{(k.)}(X)_G = \{L \in G : L \cap X \text{ dominates a cycle of type } (k.)\}$$

it has expected codim $k(c - \lambda) - r$ (make most sense if L is 1-dimensional, i.e. $\lambda = 0$)
There are many examples where these are ill-behaved.

Generic expectations

However, let

$$X_k^\Lambda = Sec_k(X)_G \cap \Sigma_\Lambda \subset \bar{X},$$

$$X_k^\lambda = X_k^\Lambda \text{ for generic } \Lambda$$

Likewise for $X_{(k.)}^\Lambda, X_{(k.)}^\lambda$.
There are no bad examples of these. We expect they are well-behaved.

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Known results

Classical results: for curves, surfaces, X_k^λ well behaved.
Mather- Alzati-Ottaviani: $X_{(k)}^0$ well behaved (smooth of expected dimension)
Other results on X_k^0: Beheshti- Eisenbud et al.
Definitive and optimal result on $X_{(k.)}^0$: Gruson-Peskine (2013):
$X_{(k.)}^0$ smooth of expected dimension at a point L such that $L \cap X$ is strictly of type $(k.)$.
X_k^0 is smooth at L whenever $\mathrm{length}(L \cap X) = k$.
Consequently, a general secant of type $(k.)$ is strict.

Secant rational curves

Result 1: General ambient spaces

X= smooth subvariety of smooth variety P

G=parameter space for nice family of rational curves on P, filling up P.

Can define $Sec_k(X)_G, Sec_(k.)(X)_G$ similarly as the curves in G secant to X.

Then also $X^{\Lambda}_{k,G}$= curves in G k-secant to X and passing through Λ, $X^0_{k,G} = X^{\lambda}_{k,G}$ for Λ general.

Similarly for $X^0_{(k.),G}$.

Theorem

$X^0_{(k.),G}$ is smooth of expected dimension near any L such that $L \cap X$ is strictly of type $(k.)$.

Near any L, $X^0_{(k.),G}$ has the expected singularities (in particular, smooth normalization).

Singularities

'Expected singularities':
if $L \cap X$ has length $k = \sum k_i$
$=$ singularities of space of divisors of type $(k.)$ on $\mathbb{A}^1$;
=space of polynomials factorizable as $\prod (x - a_i)^{k_i}$
smooth if $(k.) = (1^k)$.
if $L \cap X$ has length $> k$:
$=$ union of branches as above, corresponding to length-k subschemes of $L \cap X$.

Example

$P = G(2,5) = \mathbb{G}(1,4), X \subset P$ a 4-fold.
P contains a 6-dimensional family of 'lines' (i.e. pencils) For any $\Lambda \in P$, there is a 3-dimensional family of these lines through Λ. Of these, finitely many will be 'trisecant' to X, provided Λ is general.

Curvilinear case

Now back to projective space. G = Grassmannian of $(\lambda+1)$-spaces.
Result 2: curvilinear fibres.
Now λ is arbitrary but we assume $L \cap X$ is curvilinear (=embedding dimension 1 or less).
This happens always when $\lambda = 0$ but also for $\lambda > 0$ in small dimensions: for example, whenever

$$\lambda < \min(c, c+1-n/2).$$

Theorem

$X^{\lambda}_{(k.)}$ is smooth of expected dimension near any L such that $L \cap X$ is strictly of type $(k.)$.
Near any $L \in G$ with $L \cap X$ finite, $X^{0}_{(k.)}$ has the expected singularities (in particular, smooth normalization).

Superficial case

Result 3: superficial fibres, no contact conditions.
Here we assume $L \cap X$ has local embedding dimension 2 or less.
This happens provided $\lambda \leq 1$ but also for other λ when dimensions are small.

Theorem

Assume L is general in X_k^λ and $c > \lambda + 1$.
Then $Z = L \cap X$ is reduced. Moreover Z maps to a transverse k-fold point (= transverse tangent planes) of $\bar{X}$.

For example, the hypotheses in the Thm are satisfied for $\lambda = 2, c = 4$ provided $\dim(X) \leq 11$.

Remark

Essential ingredient: smoothness of Hilbert scheme of points on a smooth surface
Hence, smoothness of Hilbert scheme at any suprficial scheme.
Expect: Same conclusion holds whenever $L \cap X$ is unobstructed as abstract scheme.

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Sketch of proof: Uniformity principle

Idea of proof in simplest case:
$\lambda = 0, P = \mathbb{P}^N$.
Based on 'Uniformity principle' (Gruson-Peskine):
Project to $\mathbb{P}^{N-1}$ from $\Lambda \in \mathbb{P}^N$ generic.
L general k-secant through Λ, $x_1, ..., x_k \in L \cap X$.
Then, pick an *arbitrary* point $x \in L \setminus \{x_1, ..., x_k\}$.
The projection is unramified at x. In other words, x moves freely as L moves.
In particular, $x \notin X$.

More uniformity

Quiz: what's another, very common, 'uniformity principle' in Algebraic Geometry ?
Answer: a vector bundle on $\mathbb{P}^1$ that is generically generated by sections is a direct sum of nonnegative line bundles, hence is everywhere generated (and as a bonus, has $H^1 = 0$)
H^1 is related to obstruction theory, so this suggests setting up the problem as a deformation theory problem.

Secant sheaf

Let N denote the normal bundle of L in $\mathbb{P}^N$
$(= (N-1)\mathcal{O}(1)$ in case $\lambda = 0)$.
Can define a *secant subsheaf* $N^s \subset N$ corresponding to local motions of L preserving the total length of $L \cap X$.
Explicitly,

$$N^s = \{\phi \in Hom(\mathcal{I}_L, \mathcal{O}_L) : \phi(\mathcal{I}_X \cap \mathcal{I}_L) \subset \mathcal{I}_X.\mathcal{O}_L = \mathcal{I}_X/(\mathcal{I}_X \cap I_L)\}$$

It has colength $k(c-1)$ in N.
Now, our assumption that Λ was general in $\mathbb{P}^N$ means that N^s is generated by sections at Λ, hence generically, hence everywhere.
Hence $H^1(N^s) = 0$ so deformations are unobstructed, end of story.

Higher dimension

If $\lambda > 0$, so $L \simeq \mathbb{P}^{\lambda+1}$ is no longer $\mathbb{P}^1$, use instead the fact that N^s coincides with N_L over the hyperplane $\Lambda \subset L$ (because $\Lambda \cap X = \emptyset$), plus the fact that sections of N^s 'move with Λ', i.e. given any section of N_Λ, there is a section of N^s compatible with it (by generality of Λ), to conclude

$$H^1(N^s(-1)) = 0.$$

Hence we have unobstructedness and, moreover, the sheaf N^s is 0-regular.
Moreover, this argument also applies to suitable subsheaves of N^s.
This can be used to prove the result on superficial fibres, using smoothness of the Hilbert scheme at any superficial scheme Z.

Transversality

Once we know Z is reduced, we have

$$N_L/N^s = \bigoplus_{p \in Z} N_{L,p}/(T_{X,p} \cap T_{L,p}) =: \bigoplus Q_p$$

therefore the surjectivity of $H^0(N_L(-1)) \to \bigoplus Q_p(-1)$
(by the vanishing of $H^1(N^s(-1))$)
yields the transversality of the branch tangent spaces to $\bar{X}$ at $\pi_\Lambda(Z)$.

Open question

Big open question:
For $\lambda > 1$, can a nonsmoothable finite scheme occur in a fibre of a generic projection from a λ-plane ?