A relation between the moduli space of some irreducible holomorphic symplectic fourfolds and the moduli space of cubic threefolds

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Introduction

- The main objects of my talk are irreducible holomorphic symplectic manifolds and automorphisms.
- Connection with the moduli space of smooth cubic threefolds (work of Allcock-Carlson-Toledo, 2011).
- joint work with S. Boissière and C. Camere.

Definition

A complex manifold X is an irreducible holomorphic symplectic manifolds (IHS for short) if it is a smooth, compact, Kähler manifold such that

- X is simply connected;
- X admits a unique (up to scalar multiplication) holomorphic 2-form $\omega_X \in H^0(\Omega^2_X)$ wich is everywhere non–degenerate.

- Consequences:
  - $\dim X = 2n$ is even,
  - the canonical divisor K_X is trivial.

Examples

- 2 well known families of examples (Beauville-Fujiki):
  - The Hilbert scheme parametrizing n points on a K3 surface S. We denote it by $S^{[n]}$, $\dim S^{[n]} = 2n$.
  - The generalized Kummer varieties K_{n-1} of dimension $2(n-1)$ (for $n = 2$ these are the Kummer surfaces).

Moduli space

- We recall the description of the moduli space of $S^{[n]}$, $n \geq 2$.
- Recall that

$$(H^2(S^{[n]}, \mathbb{Z}), q) = U^3 \oplus E_8^2 \oplus \langle -2(n-1) \rangle := \mathcal{L},$$

with q the Beauville-Bogomolov-Fujiki quadratic form, U the hyperbolic lattice of rank 2 and signature $(1,1)$, E_8 the negative definite lattice associated to the corresponding Dynkin diagram.

- There is a surjective holomorphic map (Huybrechts, Markmann, Verbitsky):

$$\mathcal{P} : \mathcal{M}^0_{\mathcal{L}} \longrightarrow \Omega_{\mathcal{L}} = \{[\omega] \in \mathbb{P}(\mathcal{L} \otimes \mathbb{C}) \,|\, q(\omega) = 0,\ q(\omega, \bar{\omega}) > 0\}$$

defined by $\mathcal{P}((X, \omega_X, \eta_X)) = \eta_X^{-1}(\omega_X)$.

Where $\mathcal{M}^0_{\mathcal{L}}$ is a connected component of the moduli space of manifolds X that are deformation equivalent to $S^{[n]}$ (we write $X \sim S^{[n]}$ or say that X is of type $S^{[n]}$) and

$$\eta_X : \mathcal{L} \longrightarrow H^2(X, \mathbb{Z}),\ \text{is a marking}$$

- It is a 21-dimensional analytic open space.

Automorphisms

Definition

Let $X \sim S^{[n]}$. An automorphism $\sigma \in \mathrm{Aut}(X)$ of finite order m is symplectic if $\sigma^*\omega_X = \omega_X$, and it is (purely) non-symplectic if $\sigma^*\omega_X = \zeta_m \omega_X$, where ζ_m is a primitive mth-root of unity.

- The automorphism σ induces an action on $H^2(X, \mathbb{Z})$.
- Denote

$$\begin{array}{rcl} T & = & \{x \in H^2(X,\mathbb{Z}) \mid \sigma^*(x) = x\}, \text{ the invariant lattice,} \\ S & = & T^\perp \cap H^2(X,\mathbb{Z}), \text{ the orthogonal complement of } T. \end{array}$$

- Assume σ acts on X (purely) non-symplectically.
- Let

$$\begin{aligned}\mathrm{NS}(X) &= H^{1,1}(X) \cap H^2(X,\mathbb{Z}), \text{ the Néron Severi lattice},\\ \mathrm{Transc}(X) &= NS(X)^{\perp} \cap H^2(X,\mathbb{Z}), \text{ the transcendental lattice}.\end{aligned}$$

- Then $T \subset \mathrm{NS}(X)$ and $\mathrm{Transc}(X) \subset S$.

Examples

- Natural automorphisms: φ acts non-symplectically on a K3 surfaces S, induces $\varphi^{[n]}$ acting non-symplectically on $S^{[n]}$.
- Fano variety of lines:

$$V = \{x_0^3 + f_3(x_1, \ldots, x_5) = 0\} \subset \mathbb{P}^5 \text{ smooth cubic fourfold}$$
$$F(V) := \{l \in \text{Grass}(1,5) | l \subset V\}, \text{ Fano variety of lines of V.}$$

- Beauville-Donagi '85: $F(V) \sim S^{[2]}$.
- The automorphism of $\mathbb{P}^5$

$$\sigma : (x_0 : x_1 : \ldots : x_5) \mapsto (\zeta_3 x_0 : x_1 : \ldots : x_5),\ \zeta_3 = \exp^{\frac{2\pi i}{3}}$$

induces a non-symplectic automorphism of order 3 on $F(V)$.

In this example (Boissière-Camere-S. 2015):

$$T = \langle 6 \rangle; \;\; S = U^2 \oplus E_8^2 \oplus A_2$$

where

$$A_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

so that $\operatorname{sgn} S = (2, 20)$.

Setting

- In this talk we consider X an IHS manifold that is deformation equivalent to $S^{[2]}$, i.e. the dimension is 4.
- We have $H^2(S^{[2]}, \mathbb{Z}) = U^3 \oplus E_8^2 \oplus \langle -2 \rangle$
- σ acts on X non-symplectically with prime order $p \neq 2$, i.e. $\sigma^* \omega_X = \zeta_p \omega_X$.
- Recall that $2 \leq p \leq 23$ (and all cases occur!).

Aim of the talk

- We want to study the moduli space of pairs (X, σ).
- We want to generalize a similar construction of van Geemen, Dolgachev, Kondo for K3 surfaces and of Joumaah for involutions on $S^{[2]}$.

The construction

Definition

Let T be an even, non–degenerate lattice of signature $(1, r-1)$, $r = \operatorname{rank} T$.
A *T-polarized IHS manifold* deformation equivalent to $S^{[2]}$ is a pair (X, ι) such that

- $X \sim S^{[2]}$,
- $\iota : T \hookrightarrow NS(X)$ is a primitive embedding.

- Consider now $X \sim S^{[2]}$ and σ acting non–symplectically on it with order p.
- We get in a natural way a morphism:

$$\rho : \langle \sigma \rangle \longrightarrow O(\mathcal{L})$$

from the cyclic group of order p generated by σ to the isometries of the lattice $\mathcal{L} = U^3 \oplus E_8^2 \oplus \langle -2 \rangle$.

(ρ, T)-polarizations

Definition

Let (Y, η_Y) with $Y \sim S^{[2]}$ and marking $\eta_Y : \mathcal{L} \longrightarrow H^2(Y, \mathbb{Z})$. A (ρ, T)-polarization of (Y, η_Y) is:

- A T-polarization of Y compatible with the marking η_Y.
- An action by a non–symplectic automorphism σ_Y of order p on Y such that the action of σ_Y on $H^2(Y, \mathbb{Z})$ is the "same" as the action of σ, i.e. $\sigma_Y^* = \eta_Y \rho(\sigma) \eta_Y^{-1}$.

- Observe that the second condition ensures that the invariant lattice for the action of σ_Y on $H^2(Y, \mathbb{Z})$ is isometric to T.
- One can define in a natural way isomorphism classes of (ρ, T)-polarized IHS manifolds.

Period map

- For a (ρ, T)-polarized IHS manifold (Y, σ_Y) as before recall that $T \subset \mathrm{NS}(Y)$ and $\mathrm{Transc}(Y) \subset T^{\perp} \cap \mathcal{L} = S$.
- More precisely the period $\omega_Y \in S_{\zeta_p}$ (the eigenspace of σ on $S \otimes \mathbb{C}$ with respect to the eigenvalue ζ_p).
- We have

$$[\omega_Y] \in \mathbb{B} := \{[\omega] \in \mathbb{P}(S_{\zeta_p} \otimes \mathbb{C}) \,|\, q(\omega, \bar{\omega}) > 0\}$$

- Since we are assuming $p \neq 2$ we get the condition $q(\omega) = 0$ for free.

A complex ball

- Let m be the multiplicity of the eigenvalue ζ_p of σ_Y on $S \otimes \mathbb{C}$. We have $(p-1)m = \operatorname{rank} S$.
- The quadratic form q restricted to S_{ζ_p} is an hermitian form of signature $(1, m-1)$.
- $\mathbb{B}$ is isomorphic to a complex ball of dimension $m-1$.
- It is not true that for any $\omega \in \mathbb{B}$ we can find an IHS manifold $Y \sim S^{[2]}$ with non–symplectic automorphism of order p of the fixed type (i.e. such that Y is (ρ, T)–polarized).

MBM classes

Definition

Let $X \sim S^{[2]}$ and $\delta \in H^{1,1}(X, \mathbb{Q})$ a non–zero class, $q(\delta) < 0$. We call δ a MBM class if there exists a birational map

$$F : X \dashrightarrow Y$$

and $g \in O^+(H^2(X, \mathbb{Z}))$ that is a Hodge isometry, such that a face of the Kähler cone $g(f^*(\mathcal{K}_Y))$ is contained in $\delta^\perp$.

- Here $O^+(H^2(X, \mathbb{Z}))$ denotes the isometries of $H^2(X, \mathbb{Z})$ which preserve the positive cone.
- We denote by $\Delta(X)$ the set of MBM classes that are primitive and integral, i.e. they belong to $\mathrm{NS}(X)$.

Amerik-Verbitsky 2014

Let $X \sim S^{[2]}$ and

$$\mathcal{C}_X \subset \{\alpha \in H^{1,1}(X,\mathbb{R}) \,|\, q(\alpha) > 0\}$$

The positive cone, i.e. the component that contains the Kähler cone $\mathcal{K}_X$.

Theorem (Amerik-Verbitsky 2014)

The Kähler cone $\mathcal{K}_X$ is a connected component of

$$\mathcal{C}_X \setminus \bigcup_{\delta \in \Delta(X)} \delta^{\perp}$$

- The MBM classes play the role of (-2)-classes for K3 surfaces.
- The connected components in the previous decomposition correspond to the birational models of X (finite number).

Our theorem

Same assumption as above with $X \sim S^{[2]}$ and σ acting non–symplectically on it with order p, recall that S_{ζ_p} is the eigenspace corresponding to the eigenvalue ζ_p for the action of σ on $S \otimes \mathbb{C}$.

Theorem (Boissière-Camere-S. 2015)

Assume $\dim S_{\zeta_p} \geq 2$ *then*

- *If* (Y, η_Y) *is an IHS manifold of type* $S^{[2]}$ *which is* (ρ, T)*-polarized then the period belongs to*

$$\mathbb{B} \backslash \Delta$$

  where $\Delta = \bigcup_{\delta \in \Delta(S)} \delta^{\perp}$, *and* $\Delta(S)$ *are the classes in* S *that are primitive integral MBM classes for some deformation of* X.
- *The converse is also true!*

- In the second part of the theorem we can have several birational models in the fiber over a point $\omega \in \mathbb{B}\backslash\Delta$.
- If $\omega \in \Delta$ by the surjectivity of the period map we have $Y \sim S^{[2]}$ with $\omega_Y = \omega$ but Y does not admit a non–symplectic automorphism of the fixed kind (i.e. such that Y is (ρ, T)–polarized).

Fano variety of lines

- Recall the example

$$V = \{x_0^3 + f_3(x_1,\dots,x_5) = 0\} \subset \mathbb{P}^5 \text{ smooth cubic fourfold,}$$
$$F(V) := \{l \in \mathrm{Grass}(1,5) | l \subset V\}, \text{Fano variety of lines of V}$$

- The automorphism of $\mathbb{P}^5$

$$\sigma : (x_0 : x_1 : \dots : x_5) \mapsto (\zeta_3 x_0 : x_1 : \dots : x_5),\ \zeta_3 = \exp^{\frac{2\pi i}{3}}$$

induces a non-symplectic automorphism of order 3 on $F(V)$.

- We have

$$T = \langle 6 \rangle,\ S = U^2 \oplus E_8^2 \oplus A_2,$$

$\operatorname{sgn} S = (2, 20)$.

The Ball quotient

- In this case S_{ζ_3} is 11-dimensional and the moduli space of pairs $(F(V), \sigma)$ can be identified with the 10-dimensional ball quotient:

$$\Omega := \frac{\mathbb{B}\backslash\Delta}{\Gamma}$$

where Γ is an arithmetic subgroup of the group of isometries of $\mathcal{L}$.

- Taking a generic $\omega \in \Omega$ we get over this point only one IHS manifold of type $S^{[2]}$ with non–symplectic automorphism of order three and invariant lattice equal to $\langle 6 \rangle$. This is $(F(V), \sigma)$ (number of moduli of such pairs is 10).

Cubic threefolds

Recall that a smooth cubic threefold in $\mathbb{P}^4$ depends also on 10 parameters.....

Allcock-Carlson-Toledo 2011

- The moduli space of smooth cubic threefolds is exactly

$$\Omega := \frac{\mathbb{B}\backslash\Delta}{\Gamma}$$

- A rich geometry....

Ingredients in the proof of the theorem

- Result of Amerik-Verbitsky on the decomposition of the positive cone.
- Markman-Verbitsky Torelli theorem for $X \sim S^{[2]}$:
  Let $\bar{f} \in O^{+}(H^2(X,\mathbb{Z}))$ (isometries that preserve the positive cone) which is also an Hodge isometry and assume that f preserves the Kähler cone. Then there exists a unique automorphism $f : X \longrightarrow X$ such that $f^* = \bar{f}$.
- Methods of Joumaah for a similar description of the moduli space of IHS manifolds with non–symplectic involution.