Flag bundles and homogeneity of Fano manifolds

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joint with
G. Occhetta, K. Watanabe and J. Wiśniewski

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Abstract

- Only known examples of X Fano with T_X nef are RH spaces.
- Strategy: using rational curves contained in X to construct a flag manifold dominating X.
- Goal: illustrate this method by proving the homogeneity of manifolds whose family of minimal rational curves satisfies certain homogeneity conditions at every point.
- Material belongs to joint works with G. Occhetta, K. Watanabe, and J. Wiśniewski (available online here and here)

Framework

- X smooth complex projective variety
- *Fano manifold*: $-K_X = \det(T_X)$ ample
- *RH manifold* ($\mathbb{P}^n$, quadrics, Grassmannians,...):

$$\frac{G}{P} = \frac{\text{Semisimple}}{\text{Parabolic}}$$

Contents

Contractions of Fano manifolds

- $N_1(X)$: real lin. comb. of irred. curves mod $\sim_{\mathrm{num}}$
- $\rho_X := \dim(N_1(X))$: Picard number of X
- $\overline{NE(X)} \subset N_1(X)$: cl. of the cone gen. by eff. 1-cycles

Contractions of Fano manifolds

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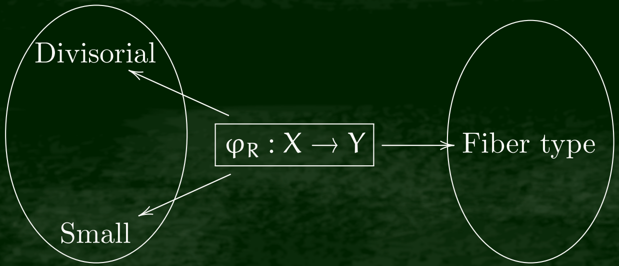
Theorem (Cone and contraction theorem for Fanos)

If X Fano:

- $\overline{NE(X)} = NE(X)$ *rational polyhedral.*
- *For every extremal ray R de $\overline{NE(X)}$, there exists $\varphi_R : X \to Y$ contracting curves with class in R.*
- *R gen. by the class of a rational curve.*

Types of contractions

- 3 types:



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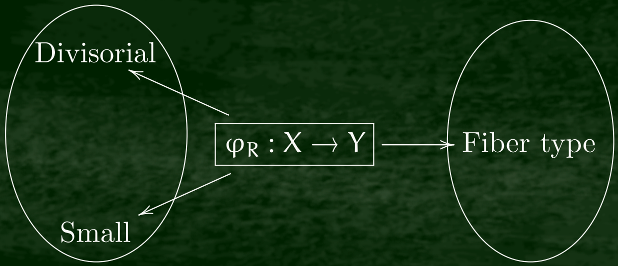


Problem
Find (nontrivial) examples of Fanos with only smooth f.t. contractions.

- (Picard number one, products) Rational homogeneous, ...
- Examples usually have large automorphism groups (why?)

Contractions of RH manifolds

G semisimple, $\mathfrak{g}$ Lie algebra, $\mathcal{D}$ Dynkin diagram (nodes D)

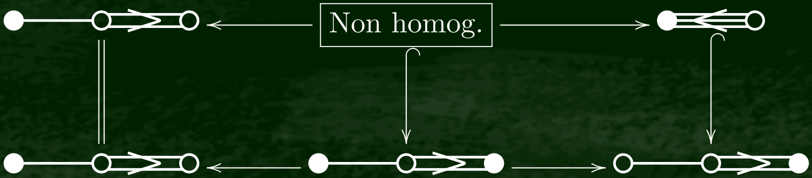
- (marking $\iff$ RH manifold) $\mathrm{I} \subset \mathrm{D} \quad \mapsto \quad \mathcal{D}(\mathrm{I}) = \mathsf{G}/\mathsf{P}(\mathrm{I})$
- (contractions $\iff$ delete nodes) $\mathrm{I} \supset \mathrm{J} \quad \mapsto \quad \mathsf{p} : \mathcal{D}(\mathrm{I}) \to \mathcal{D}(\mathrm{J})$
- Fibers of p are homogeneous: $\mathcal{D}_{\mathrm{J}}(\mathrm{I} \setminus \mathrm{J})$
- $\mathrm{NE}(\mathcal{D}(\mathrm{I}))$ is simplicial, generated by the marked nodes I of D
- For complete flag $\mathcal{D}(\mathrm{D})$, elem. cont. are $\mathbb{P}^1$-bundles $(\mathsf{p}_1, \ldots, \mathsf{p}_n)$
- Lie algebra $\mathfrak{g} \quad \leftrightarrow \quad$ intersection matrix $(-\mathsf{K}_i \cdot \Gamma_j)$

Characterization of complete flags [OSW]

Theorem
X *smooth projective variety, such that*
$\exists\ \Gamma_i \in N_1(X)$, $i = 1, \ldots, \rho_X$ *extremal* K_X*-negative classes, whose associated contractions are smooth* $\mathbb{P}^1$*-bundles.*
Then $X \cong G/B$, *for some* G *semisimple.*

A non homogeneous example

- (Ottaviani, Kanemitsu)



$$\begin{tikzcd}
\mathbb{Q}^5 \arrow[d, equal] & \mathbb{P}(\mathcal{E}) \arrow[l, "\mathbb{P}^2"'] \arrow[r, "\mathbb{P}^2"] \arrow[d, hook] & \mathbb{Q}^5 \arrow[d, hook] \\
\mathbb{Q}^5 & \mathrm{B}_3(1,3) \arrow[l, "\mathbb{P}^3"'] \arrow[r, "\mathbb{P}^2"] & \mathbb{Q}^6
\end{tikzcd}$$

Contents

Positivity and homogeneity

Theorem (Mori'79)

T_{X} *ample* $\Rightarrow \mathsf{X} = \mathbb{P}^{\mathsf{m}}$.

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- T_X nef $\Rightarrow$??
- Examples:

$$\begin{tikzcd} & \text{Abelian} \\ \text{Homog. mfds.} \arrow[ur] \arrow[dr] & \\ & \text{RH} \end{tikzcd}$$

Positivity and homogeneity

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Theorem (Demailly–Peternell–Schneider'94)

$$\mathsf{T}_{\mathsf{X}} \ nef \Rightarrow \begin{cases} \mathsf{X} \xleftarrow{\textit{étale}} \mathsf{X}' \xrightarrow{\mathsf{F}} \mathsf{A} \\ \mathsf{F} \ \textit{Fano}, \ \mathsf{T}_{\mathsf{F}} \ \textit{nef} \end{cases}$$

The CP Conjecture

Conjecture (Campana–Peternell'91)
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For X CP manifold:

- Every contraction of X is smooth f.t.
- $\pi : X \to Y$ contraction $\Rightarrow$ $Y, \pi^{-1}(y)$ are CP manifolds ([DPS,SW])
- NE(X) is *simplicial*

Evidences

- ☑ $\dim = 3$ [CP'91]
- ☑ $\dim = 4$ [CP'93, Mok'02, Hw'06]
- ☑ $\dim = 5$ [Wa'12, Ka'15]
- ☑ T_X big and 1-ample [SW'04]
- ☑ T_X is strongly Griffiths nef [Y'14]
- ☑ X horospherical [L'15]

Bottom-up arguments: example

- G Lie group, $P = P(I)$
- Reconstruct G/B from G/P as successive *families* of rat. curves

$\mathbb{P}(T_{\mathbb{P}^n}) = A_n(1, n)$

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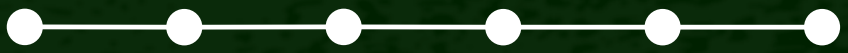

$A_n(1, 2, 3, 4, \ldots, n)$

- Sometimes: no complete families

Bottom-up arguments: example

- G Lie group, $P = P(I)$
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$A_n(1, 2, 3, 4, \ldots, n)$

- Sometimes: no complete families
- In progress

Bottom-up arguments

- *Width* of a CP manifold X:

$$\tau(X) := \sum_{\substack{C \text{ min.} \\ \text{rat. curve}}} (-K_X \cdot C - 2) \in \mathbb{Z}_{\geq 0}$$

- CP conjecture holds for FT manifolds is equivalent to:

Conjecture (Untangling)

X *CP*, $\tau(X) > 0$. $\exists$ *contr.* $X_1 \to X$, *with* X_1 *CP*, $\tau(X_1) < \tau(X)$.

Contents

VMRT of a family of rational curves

- Family of minimal rational curves:

$$\begin{tikzcd} & \mathcal{U} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{M} & & X \end{tikzcd}$$

- $\tau_x : \mathcal{M}_x \to \mathbb{P}(\Omega_{X,x})$ bir. fin. morphism (x gen.) [HM],[K]
- VMRT at x: $\mathcal{C}_x = \tau_x(\mathcal{M}_x)$

The theorems of Mok and Hong-Hwang

Theorem
X be a Fano manifold, $\rho_X = 1$, $x \in X$ general.
G/P corresponding to a long simple root, $o \in G/P$.
$\mathcal{C}_o \subset \mathbb{P}(\Omega_{G/P,o})$, $\mathcal{C}_x \subset \mathbb{P}(\Omega_{X,x})$ the corresponding VMRT's. Then:

$$\mathcal{C}_x \overset{proj.}{\cong} \mathcal{C}_o \Rightarrow X \cong G/P$$

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$\mathcal{C}_o \subset \mathbb{P}(\Omega_{G/P,o})$, $\mathcal{C}_x \subset \mathbb{P}(\Omega_{X,x})$ *the corresponding VMRT's. Then:*

$$\mathcal{C}_x \overset{proj.}{\cong} \mathcal{C}_o \Rightarrow X \cong G/P$$

- VMRT of G/P: not homogeneous in general
- Within the class of G/P's, Picard one, associated to long simple roots, the VMRT is homogeneous and determines G/P

Recognizing homogeneity from rational curves

Theorem (OSW)

Let X *be a Fano manifold,* $\rho_X = 1$*,*
$p : \mathcal{U} \to \mathcal{M}$ *be an unsplit dominating complete family of rational curves with smooth evaluation morphism* $q : \mathcal{U} \to X$*.*
Assume that $\mathcal{M}_x \cong G/P$*, for every* $x \in X$*.*
Then X *is RH.*

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Assume that $\mathcal{M}_x \cong G/P$*, for every* $x \in X$*.*
Then X *is RH.*

- No need to consider any particular projective embedding of $\mathcal{M}_x$
- Need to assume that q is smooth
- Need to assume that *every* $\mathcal{M}_x$ is RH (unless rel. Pic. 1)

Sketch of the proof

1. Reconstructing the complete flag:

$$\begin{tikzcd} & \widetilde{\mathcal{U}} \arrow[d, "\pi"'] \arrow[dr, "\text{rel. flag}"] & \\ \mathcal{M} & \mathcal{U} \arrow[l, "p"'] \arrow[r, "q"] & X \end{tikzcd}$$

2. Finding an extra $\mathbb{P}^1$-bundle:

$$\begin{tikzcd} \widetilde{\mathcal{M}} \arrow[d] & \widetilde{\mathcal{U}} \arrow[l, "\text{lift}"] \arrow[d, "\pi"'] \arrow[dr, "\text{rel. flag}"] & \\ \mathcal{M} & \mathcal{U} \arrow[l, "p"'] \arrow[r, "q"] & X \end{tikzcd}$$

3. Use the characterization of flags for $\widetilde{\mathcal{U}} \Rightarrow X$ RH

1. Reconstructing the complete flag

- Fiber F (RH), then $q : \mathcal{U} \to X$ is locally (analytically) trivial ([FG])
- q provides $\theta \in H^1(X, G)$, $G = \mathrm{Aut}(F)^\circ$ (semisimple), $F = G/P$
- θ determines a principal bundle E_G over X such that $\mathcal{U} = E_G \times_G G/P$
- Set $\widetilde{\mathcal{U}} := E_G \times_G G/B \to X$
- Comes with a contraction to $\pi : \widetilde{\mathcal{U}} \to \mathcal{U}$
- $\mathbb{P}^1$-bundle structures: $\pi_i : \widetilde{\mathcal{U}} \to \mathcal{U}_i$ factoring π ($i \in I$)
- $\mathbb{P}^1$-bundle structures: $\pi_j : \widetilde{\mathcal{U}} \to \mathcal{U}_j$ not factoring π ($j \in J$)
- $\sharp(I \cap J) = \rho_{\widetilde{\mathcal{U}}} - 1$

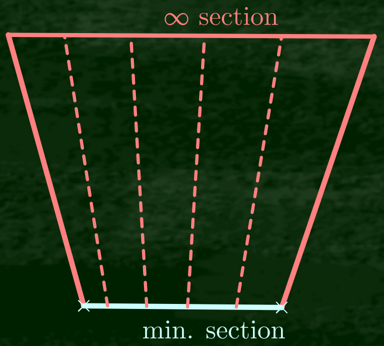
Interlude: Grothendieck's theorem

Theorem
For the Riemann sphere $\mathbb{P}^1$*,*
G *reductive,* $\mathsf{H} \subset \mathsf{G}$ *Cartan,* W *Weyl group :*

$$\mathsf{H}^1(\mathbb{P}^1, \mathsf{H})/\mathsf{W} \cong \mathsf{H}^1(\mathbb{P}^1, \mathsf{G}).$$

- We are interested in G semisimple
- A cocycle $\theta \in \mathsf{H}^1(\mathbb{P}^1, \mathsf{G})$ provides a G/B-bundle $\widetilde{\mathsf{E}} \to \mathbb{P}^1$

A $\mathbb{P}^1$-bundle



$G = PGl_2$

$H \mapsto$ choice of $C_0,\ C_\infty$

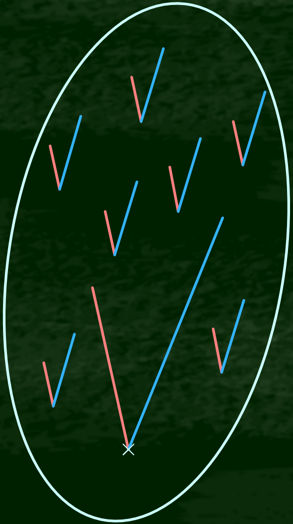
corresp. Borel subgps. $B_0, B_\infty \supset H$

invariant $e = K_{\mathrm{rel}} \cdot C_0 = \phi(\theta)$

$\phi : H^1(X, H) \to L \cong \mathbb{Z}$

e even $\iff$ Sl_2 –bundle

A complete flag



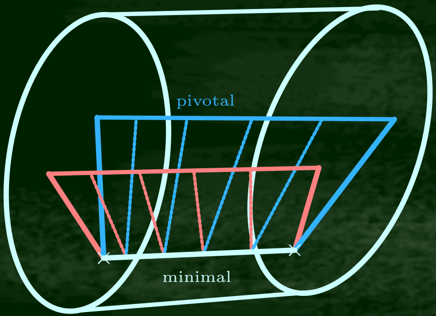
K_t rel. can. $G/B \to G/P_t$ ($\mathbb{P}^1$-bundles)

$\{\alpha_i\}$ simple roots of G

$$K_{G/B} = \sum_{t=1}^{k} b_t K_t \text{ where}$$

$$\sum_{\beta \in \Phi^+} \beta = \sum_{t=1}^{k} b_t \alpha_t$$

A flag bundle



$H \subset G$(Cartan, semisimple)

$\pi : E \to \mathbb{P}^1$ flag bundle of type G

$H \mapsto$ choice of $\sharp(W)$ sections

corresp. Borel subgps. B_w, $w \in W$

one B corresp. to min. section C

simple roots α_t, invariants d_t

E det. by Dynkin diag

tagged by the d_t's

$$K_{rel} = \sum_{t=1}^{k} b_t K_t$$

2. An extra contraction (I)

- In our situation:

$$\begin{tikzcd} \boxed{??} \arrow[d] & \widetilde{\mathcal{U}} \arrow[l, "\text{lift}"'] \arrow[d] \arrow[dr, "\text{rel. flag}"] & \\ \mathcal{M} & \mathcal{U} \arrow[l, "p"'] \arrow[r, "q"] & X \end{tikzcd}$$

- Take a curve $\mathbb{P}^1$ of the family $\mathcal{M}$:

$$\begin{tikzcd} s^*\widetilde{\mathcal{U}} \arrow[r] \arrow[d] & \mathbb{P}^1 \arrow[d, hook, "s"] & \\ \widetilde{\mathcal{U}} \arrow[r, "\pi"] & \mathcal{U} \arrow[r, "q"] & X \end{tikzcd}$$

- $s^*\widetilde{\mathcal{U}}$ is a G_I/B_I-bundle over $\mathbb{P}^1$: enough to show it is trivial
- $(d_1, \ldots, d_k)$ tag of $s^*\pi^*\widetilde{\mathcal{U}}$: enough to show $d_i = 0$ for $i \in I$

2. An extra contraction (II)

$$\begin{tikzcd} & \mathbb{P}^1 \arrow[dl, "\widetilde{s}"'] \arrow[d, "s"] & \\ \widetilde{\mathcal{U}} \arrow[r, "\pi"] \arrow[rr, bend right, "\widetilde{\pi}"'] & \mathcal{U} \arrow[r, "q"] & X \end{tikzcd}$$

- $s^*\pi^*\widetilde{\mathcal{U}}$ G/B-bundle, $K_{\widetilde{\pi}} \cdot \widetilde{s}(\mathbb{P}^1) = \sum_{t \in I \cup J} b_t d_t$
- $s^*\widetilde{\mathcal{U}}$ G_I/B_I-bundle, $K_{\widetilde{\pi}} \cdot \widetilde{s}(\mathbb{P}^1) = \sum_{t \in I} c_t d_t$
- $K_q \cdot s(\mathbb{P}^1) = \sum_{i \in I} (b_i - c_i) d_i + \sum_{j \in J} b_j d_j$
- LHS$=\dim(q)$, $b_i > c_i$, $d_j > 0$, $\sum b_j \geq \dim(q) \Rightarrow d_i = 0$

Contents

Remaining cases?

Recognizing symplectic Grassmannians

Theorem (OSWa)

X *Fano manifold,* $\rho_X = 1$.
$\mathcal{M}$ *an unsplit dominating complete family of rational curves with smooth evaluation morphism, whose tangent map* τ *is a morphism.*
Assume τ_x *proj. equiv. to* $M \subset \mathbb{P}\big(H^0(M, \mathcal{O}_M(1))\big)$, *for every* $x \in X$.
Then X *is isomorphic to a symplectic Grassmannian.*

$$(C_n(r)): \; M \cong \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^{r-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{r-1}}(1)^{2n-2r}\big), \text{ embedded by } \mathcal{O}(1)$$

THANKS FOR YOUR ATTENTION

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