Geometry of genus 8 Nikulin surfaces and rationality of moduli

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1. Nikulin surfaces in low genus

- We consider complex K3 surfaces S endowed with
  - a[1] polarization $\mathcal{C}$ of genus g,
  - a line bundle $\mathcal{M} := \mathcal{O}_S(M)$ so that $2M \sim N$
  - and N is the disjoint union of 8 copies of $\mathbf{P}^1$.
  - $< \mathcal{C}, \mathcal{M} >= 0$.
- The irreducible components of the moduli have dimension 11,
- their number and lattice theoretic characterization is known. [2]

[1] big and nef

[2] Cfr. Garbagnati-Sarti, Sarti-van Geemen and then Huybrechts book

- We have $2M \sim N_1 + \cdots + N_8$ with $N_k = \mathbf{P}^1$ and $N_i N_j = -2\delta_{ij}$.

- $N := N_1 + \cdots + N_8$ defines the double covering $\pi' : \tilde{S}' \to S$ branched on N and the commutative diagram

$$\begin{tikzcd} \tilde{S}' \arrow[r, "\nu'"] \arrow[d, "\pi'"'] & \tilde{S} \arrow[d, "\pi"'] \\ S \arrow[r, "\nu"] & \overline{S} \end{tikzcd}$$

  ν is the contraction of N [3] and $\tilde{S}$ is a minimal K3 surface.

- π is the quotient map of a symplectic involution $\iota : \tilde{S} \to \tilde{S}$ branched exactly on the even set of nodes

$$\{o_1 := \nu(N_1)\ ,\ \dots\ ,\ o_8 := \nu(N_8)\} = \mathrm{Sing}\,\overline{S}.$$

[3] Let $E_i = \pi'^{-1}(N_i)$, $i = 1 \dots 8$, then E_i is an exceptional line on the smooth surface $\tilde{S}'$. It turns out that ν' is the contraction of $E_1 + \cdots + E_8$

- $\forall g\ \exists!$ integral component whose general point $[S, \mathcal{C}, \mathcal{M}]$ satisfies

$$\mathrm{Pic}\, S = \mathbb{Z}[\mathcal{C}] \perp \mathbb{L}_S,$$

where $\mathbb{L}_S$ is generated by $\mathcal{M}, \mathcal{O}_S(N_1), \ldots, \mathcal{O}_S(N_8)$. [4]

- Definition:

We will say that $(S, \mathcal{C}, \mathcal{M})$ is a *Nikulin surface of genus* g if its moduli point is in the above mentioned component.

- Notations:
  - $\mathcal{F}_g :=$ moduli of genus g K3 surfaces $(S, \mathcal{C})$,
  - $\mathcal{F}_g^N :=$ moduli of genus g Nikulin surfaces $(S, \mathcal{C}, \mathcal{M})$.

- With a slight abuse we can say that

$$\mathcal{F}_g^N \subset \mathcal{F}_g.$$

[4] As an abstract lattice $\mathbb{L}_S$ is known as the Nikulin lattice.

- An intermediate divisor in $\mathcal{F}_g$:

$$\mathcal{D}_g := \{ \ [S, \mathcal{C}] \in \mathcal{F}_g \ / \ \exists \ \mathcal{M} \in \operatorname{Pic} S, \ < \mathcal{M}, \mathcal{C} >= 0\}$$

  so that $< \mathcal{M}, \mathcal{M} >= -4$. We assume $\mathcal{C} \otimes \mathcal{M}^{-1}$ big and nef.
- For a general $[S, \mathcal{C}] \in \mathcal{D}_g$ the element $\mathcal{M}$ is unique and

$$\operatorname{Pic} S = \mathbb{Z}[\mathcal{C}] \perp \mathbb{Z}[\mathcal{M}].$$

- Clearly:

$$\mathcal{F}_g^N \subset \mathcal{D}_g \subset \mathcal{F}_g.$$

- In low genus $\mathcal{F}_g^N$ sits in a fascinating system of relations to other geometric families. We present some done work and some work in progress about this subject. [5]
- For $g \leq 10$ it seems interesting to study Mukai constructions for a Nikulin surface.
- The unirationality of $\mathcal{F}_g^N$ is known for $g \leq 7$ [6] We prove here:

[5] The in progress part jointly with A. Garbagnati
[6] Farkas-Verra to appear in Advances of Math.

- Theorem (1)

  $\mathcal{F}_8^N$ *is rational.* [7]

- Theorem (2)

  $\mathcal{D}_8$ *is birational to* $\mathbf{P}^{14} \times \mathcal{P}_6$*.*

  - $\mathcal{P}_6$ denotes the moduli space of six unordered points of $\mathbf{P}^2$.
  - Its rationality is an unknown, apparently difficult, problem.
  - A natural question: is $\mathcal{F}_g^N$ rational for $g \leq 7$?

[7] — to appear in *K3 surfaces and their moduli* Proceedings Schirmonnikoog 2014

- There is a beautiful geometry behind theorems 1 and 2 we want to discuss during this talk.
- Further notations for $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{D}_g$, $g \geq 3$:
  - $\mathcal{H} := \mathcal{C}(-M)$ and $\mathcal{A} := \mathcal{C}(-2M)$, moreover
  - $C \in |\mathcal{C}|$, $H \in |\mathcal{H}|$, $A \in |\mathcal{A}|$, [8].
- For a general $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{F}_g^N$ the map $f_{\mathcal{C}} \times f_{\mathcal{H}}$ defines an embedding

$$S \subset \mathbf{P}^g \times \mathbf{P}^{g-2}.$$

- For a general $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{D}_g$, $\mathcal{C}$ and $\mathcal{H}$ are very ample as soon as their genus is ≥ 3.

[8] provided these linear systems are not empty.

- For a general $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{F}_g^N$ we have:
  - $f_{\mathcal{H}}(S) = S$ and $f_{\mathcal{H}}(N_i)$ is a line.
  - $f_{\mathcal{C}}(S) = \overline{S}$ and $f_{\mathcal{C}}(N_i)$ is a node.
- The next characterization is useful for $g \geq 8$:

Proposition

Let $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{D}_g$, the following conditions are equivalent:

- $h^0(\mathcal{M}) = 0$ *and* $h^0(\mathcal{M}^{\otimes 2}) = 1$,
- $\exists\ N_1 \dots N_8$ *disjoint copies of* $\mathbf{P}^1$ / $HN_i = 1$, $AN_i = 2$ *and*

$$\mathcal{O}_S(N_1 + \cdots + N_8) \cong \mathcal{M}^{\otimes 2}$$

- In particular this characterizes $\mathcal{F}_g^N$ in $\mathcal{D}_g$ for $g \equiv 0 \mod 4$.

- Finally we fix the projective models

$$S \subset \mathbf{P}^{g-2}\ ,\ \overline{S} \subset \mathbf{P}^{g}$$

respectively defined by $\mathcal{H}$ and by $\mathcal{C}$.

- We start with the geometry of $\mathcal{F}_g^N$ for $g \leq 7$.
- Omitting $g \leq 5$, we give a view on two other nice cases:

$$g = 6, 7.$$

- $g = 6$. Let $Q \subset \mathbf{P}^4$ be a smooth quadric, the tangential quadratic complex of Q is

$$W := \{l \in G(1,4) \ / \ l \text{ is tangent to } Q\}.$$

- W is endowed with the quasi-étale double covering

$$\pi : \tilde{W} \to W$$

branched on Sing W = the Veronese embedding of $\mathbf{P}^3$ in $G(1,4) \subset \mathbf{P}^9$. One can show that:

- Proposition

A general model $\overline{S}$ of genus 6 is a linear section of W.

- It follows that $\mathcal{F}_6^N$ is unirational.

- $g = 7$. Consider the model $S \subset \mathbf{P}^5$ defined by $\mathcal{H}$: S is the base locus of a net of quadrics.
- Choosing $N_1 \dots N_7$ it turns out that $C \sim C_o := R + N_1 + \cdots + N_7$, with R a rational normal quintic.
- C_o is the union of R and seven bisecant lines to it. Starting from a curve C_o, this is in the base locus S of a unique net of quadrics.
- S turns out to be a *general* Nikulin surface of genus 7 endowed with an eighth line $N_8 \sim 2C_o - 2H - N_1 - \cdots - N_7$.

- **Theorem**

The moduli space $\tilde{\mathcal{F}}_7^N$ of curves C_o is rational and has a map of degree 8

$$f : \tilde{\mathcal{F}}_7^N \to \mathcal{F}_7^N.$$[9]

[9] Actually $\tilde{\mathcal{F}}_7^N$ is the moduli of fourtuples $(S, \mathcal{C}, \mathcal{M}, N_i)$ such that $(S, \mathcal{C}, \mathcal{M})$ is a Nikulin surface of genus 7 and N_i is one of the lines in $S \subset \mathbf{P}^5$. The rationality of $\mathcal{F}_7^N$ is not clear.

2. Nikulin surfaces of genus 8 and rational normal sextics

- Let $g = 8$ and $[S, \mathcal{C}] \in \mathcal{D}_8$ be general, we have an embedding

$$S \subset \mathbf{P}^6$$

with hyperplane sections $H \sim C - M$ of genus 6.

- For $g = 8$ we have $(C - 2M)^2 = -2$ and $(C - 2M)H = 6$.

- **Proposition**

  Let $A \in |C - 2M|$ and $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{F}_8^N$ general. Then A is a smooth, integral rational normal sextic spanning $\mathbf{P}^6$. [10]

- **Proposition**

  For a general $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{F}_8^N$ the lines $N_1 \dots N_8$ are disjoint bisecant lines to A contained in S.

[10] Then the same is true by semicontinuity on $\mathcal{D}_8$.

- The Mukai-Brill-Noether theory is known for $[X, \mathcal{O}_X(1)] \in \mathcal{F}_6$ [11]:
  - CASE 1:
  - If a smooth $H \in |\mathcal{O}_X(1)|$ is not trigonal nor biregular to a plane quintic, then H is generated by quadrics.
  - $\exists!$ H-stable rank 2 vector bundle $\mathcal{E}$ on X such that:
    - (i) *det* $\mathcal{E} \cong \mathcal{O}_X(1)$;
    - (ii) $h^0(\mathcal{E}) = 5$ *and* $h^i(\mathcal{E}) = 0$ *for* $i \geq 1$;
    - (iii) *det* $: \wedge^2 H^0(\mathcal{E}) \to H^0(\mathcal{O}_{\mathbf{P}^6}(1))$ *is surjective.*

[11] For simplicity we assume that $\mathcal{O}_X(1)$ is very ample

- Let $G(1,4) \subset \mathbf{P}^9 := \mathbf{P} \wedge^2 H^0(\mathcal{E})^*$ be the Plücker embedding of the Grassmannian of lines of $\mathbf{P}H^0(\mathcal{E})^*$. Let $\mathbf{P}^6 = \mathbf{P}H^0(\mathcal{O}_X(1))^*$.

  Then the diagram

$$\begin{array}{ccc} \mathbf{P}^6 & \xrightarrow{\delta} & \mathbf{P}^9 \\ \uparrow & & \uparrow \\ X & \xrightarrow{f_{\mathcal{E}}} & G(1,4), \end{array}$$

  commutes, where $\delta := det^*$, the vertical maps are the inclusions and $f_{\mathcal{E}}$ is the embedding defined by $\mathcal{E}$:

$$x \in X \longrightarrow \mathcal{E}_x^* \subset H^0(\mathcal{E})^* \in G(1,4).$$

- Up to obvious identifications we can say that

$$X \subset T := \mathbf{P}^6 \cap G(1,4) \subset \mathbf{P}^9.$$

- Mukai theory in genus 6 says also that:

  (iv) *X is a quadratic section of T,*

- Since X is a smooth quadratic section of T, T is an integral threefold with isolated singularities.

- Actually T is a smooth Del Pezzo threefold of degree 5 if X is sufficiently general.

- ◦ CASE 2:
  - ◦ Assume H is either trigonal or biregular to a plane quintic. Then H has Clifford index 1 and it follows that:
  - ◦ there exists an integral curve $D \subset X$ such that either $DH = 3$ and $D^2 = 0$ or $DH = 5$ and $D^2 = 2$.
- A general genus 8 Nikulin surface occurs in case (1), not in (2).

- Proposition

Let $S \subset \mathbf{P}^6$ be a general Nikulin surface of genus 8 embedded by $f_{\mathcal{H}}$. Then S is a quadratic section of a threefold T as above. [12]

[12] PROOF Pic S is the orthogonal sum of rank 9 $\mathbb{Z}\mathcal{L} \oplus \mathbb{L}_S$, where $\mathbb{L}_S$ is the Nikulin lattice generated by $\mathcal{O}_S(M), \mathcal{O}_S(N_1) \dots \mathcal{O}_S(N_8)$. A standard computation we omit, shows that no divisor D exists such that $D^2 = 0$ and $DH = 3$ or $D^2 = 2$ and $DH = 5$. This excludes case (2).

- Let A and $S \subset T = \mathbf{P}^6 \cap G(1,4) \subset \mathbf{P}^9$ as above. Under the previous generality assumptions we study the restriction

$$\mathcal{E}_A := \mathcal{E} \otimes \mathcal{O}_A$$

of the Mukai bundle $\mathcal{E}$ and discuss the possible cases. Of course we have $\mathcal{E}_A = \mathcal{O}_{\mathbf{P}^1}(m) \oplus \mathcal{O}_{\mathbf{P}^1}(n)$ with $m + n = 6$.

Theorem

One has $\mathcal{E}_A \cong \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(3)$.

- Now we consider $\mathbb{P}_A := \mathbf{P}\mathcal{E}_A^*$ and the tautological map

$$u_A : \mathbb{P}_A \to \mathbf{P}^7 := \mathbf{P}H^0(\mathcal{E}_A)^*.$$

- Then

$$R := u_A(\mathbb{P}_A).$$

  is a rational normal scroll of degree 6.

- The next standard exact sequence will be crucial:

$$0 \to \mathcal{E}(-A) \to \mathcal{E} \to \mathcal{E}_A \to 0.$$

- The associated long exact sequence is the following:

$$0 \to H^0(\mathcal{E}) \to H^0(\mathcal{E}_A) \xrightarrow{\partial_A} H^1(\mathcal{E}(-A)) \to 0.$$

- In particular one has
  - $h^0(\mathcal{E}) = 5$,
  - $h^0(\mathcal{E}_A) = 8$,
  - $h^1(\mathcal{E}(-A)) = 3$. [13]

- The coboundary map $\partial_A : H^0(\mathcal{E}_A) \to H^1(\mathcal{E}(-A))$ defines a plane

$$P_A := \mathbf{P}\, Im\ \partial_A^* \subset \mathbf{P}^7.$$

[13] PROOF Since $\mathcal{E}(-A)$ is H-stable and $H(H-2A) < 0$, it follows $h^0(\mathcal{E}(-A)) = 0$. Furthermore we know that $h^i(\mathcal{E}) = 0$ for $i \geq 1$ and we have $h^1(\mathcal{E}_A) = 0$ because $m, n \geq 0$. This implies the statement.

- Let $\mathbf{P}^4 := \mathbf{P}H^0(\mathcal{E})^*$ and $\mathbb{P}_X := \mathbf{P}\mathcal{E}^*$. Dualizing the sequence and projectivizing we define the linear projection of center P_A:

$$\alpha_A : \mathbf{P}^7 \to \mathbf{P}^4 := \mathbf{P}H^0(\mathcal{E})^*.$$

- Furthermore we have the commutative diagram

$$\begin{tikzcd} \mathbf{P}^7 \arrow[r, "\alpha_A"] & \mathbf{P}^4 \\ \mathbb{P}_A \arrow[u, "u_A"] \arrow[r, "i"] & \mathbb{P}_X \arrow[u, "u_S"'] \end{tikzcd}$$

where the vertical arrows are the tautological maps. One expects that $\alpha_A(R)$ has exactly six apparent double points.

- *This is true generically for $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{D}_8$. Not along the Nikulin locus $\mathcal{F}_8^N$.*

- Let $G(1,7)$ be the Grassmannian of lines of $\mathbf{P}H^0(\mathcal{E}_A)^*$. Then the projection $\alpha_A : \mathbf{P}^7 \to \mathbf{P}^4$ defines a map

$$\lambda_A : G(1,7) \to G(1,4).$$

defined by the assignement $l \longrightarrow \alpha_A(l), \quad l \in G(1,7)$.

- The next diagram is commutative:

$$\begin{tikzcd} G(1,7) \arrow[r, "\lambda_A"] & G(1,4) \\ A \arrow[u, "f_{\mathcal{E}_A}"] \arrow[r, "i"] & S \arrow[u, "f_{\mathcal{E}}"'] \end{tikzcd}$$

3. Nikulin surfaces of genus 8 and symmetric cubic threefolds

- A symmetric cubic threefold is a cubic hypersurface
$$V := \{det(a_{ij}) = 0\} \subset \mathbf{P}^4,$$
where $a_{ij} = a_{ji}$ are linear forms.
- We assume dim $< a_{11} \dots a_{33} >= 5$ so that
$$V = Sec\ B,$$
B a rational normal quartic curve.
- The family of bisecant lines to B is a 3-Veronese embedding
$$W \subset G(1,4)$$
embedded as a congruence of class $(3,6)$.

- Since $\mathcal{E}_A$ is balanced then $\mathbb{P}_A = \mathbf{P}^1 \times \mathbf{P}^1$ and

$$R := u_A(\mathbb{P}_A) \subset \mathbf{P}^7 = \mathbf{P}H^0(\mathcal{E}_A)^*$$

is the image of $|\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1,3)|$.

- R is a rational normal sextic scroll: we fix it once at all.
- Restricting to R the top arrow of the previous diagram

$$\begin{array}{ccc} \mathbf{P}^7 & \xrightarrow{\alpha_A} & \mathbf{P}^4 \\ {\scriptstyle u_A}\uparrow & & \uparrow{\scriptstyle u_S} \\ \mathbb{P}_A & \xrightarrow{i} & \mathbb{P}_X \end{array}$$

- We obtain a linear projection

$$\alpha_A : R \to \mathbf{P}^4.$$

- α_A is a finite morphism of degree 1 onto its image. Let

$$Z \subset R$$

  be the subscheme of points where α_A is not an embedding. Then

$$\ell(Z) = 12$$

  by double point formula.

- In other words R has six apparent ordinary double points if α_A is a sufficiently general projection in $\mathbf{P}^4$.

- This is actually not the case for simple geometric reasons:
- A has 8 bisecant lines $N_1 \dots N_8 \subset S \subset G(1,4)$ in the Nikulin case,
- $\alpha_A(R)$ is the projection in $\mathbf{P}^4$ of the universal line over A. This is

$$\mathbb{P}_A := \{(x,l) \in \mathbf{P}^4 \times A \ / \ x \in l\} \subset \mathbf{P}^4 \times G(1,4),$$

- N_i parametrizes a pencil of lines in $\mathbf{P}^4$ of center say n_i,
- the fibre of $\mathbb{P}_A$ at $N_i \cap A$ is the disjoint union of two lines in $\mathbf{P}^4$ parametrized by N_i.
- Hence:

$$\operatorname{Sing} \alpha(R) \supseteq \{n_1 \dots n_8\} \ !$$

- Theorem
  - $\operatorname{Sing} \alpha_A(R)$ *is a rational normal quartic* B,
  - $\alpha_A(R)$ *is a degenerated K3 surface of genus 4:*
  - *let* $V = Sec\ B$ *then*

$$\alpha_A(R) = Q \cap V, \quad Q \in |\mathcal{I}_{B/\mathbf{P}^4}(2)|.$$

- So far A is defined by a special embedding

$$\alpha_A : \mathbf{P}^1 \to G(1,4),$$

as for every rational normal sextic

$$< A > \cap G(1,4) = T.$$

- But $A = W \cap T$, where $W = B^{[2]}$ embedded with class $(3,6)$.
- More geometry of the special embeddings $\alpha : \mathbf{P}^1 \to A \subset G(1,4)$:

- Special feature: A has a 1-dimensional family of bisecant lines

$$E_A := \{\text{lines } N \text{ such that } N \subset < A > \cap G(1,4)\}.$$

- The family E_A is an elliptic curve embedded in $B \times B$.[14]

- A defines another degenerated K3 surface of genus 6:

$$S_A = \bigcup N, \ N \in E_A.$$

- Actually $S_A \in |\mathcal{I}_{A/T}(2)|$, in particular

$$\mathrm{Sing}_A = A.$$

[14] It is embedded as a correspondence of type (2,2) in $B \times B$: for each $p \in B$ there are two bisecant lines to B. This defines the correspondence.

- The family of special embeddings α_A modulo $Aut\ G(1,4)$ is

$$|\mathcal{I}_{B/V}(2)|/Aut\ B = |\mathcal{O}_{B^{[2]}}(2)|/Aut\ \mathbf{P}^1$$

[15] that is a rational surface we will denote by

$$\Sigma.$$

- The considered Nikulin surface S belongs to $|\mathcal{I}_{A/T}(2)|$.
- A general $S' \in |\mathcal{I}_{A/T}(2)|$ is a smooth Nikulin surface.
- Proof: $S' = Q' \cap T$ and

$$Q' \cdot S_A = 2A + N'_1 + \cdots + N'_8,$$

N'_i a bisecant line to A.

[15] $|\mathcal{I}_{B/V}(2)|$ is naturally biregular to $|\mathcal{O}_{B^{[2]}}(2)|$. B is embedded in $B^{[2]}$ as the diagonal and the action is the action of $PGL(2) = Aut\ B$ on $B^{[2]}$.

- Let $\alpha \in \mathbf{P}^5 := |\mathcal{I}_{B/V}(2)|$, we denote by $\alpha : \mathbf{P}^1 \to G(1,4)$ the corresponding sextic embedding and put $A = \alpha(\mathbf{P}^1)$:

- From the previous remarks and construction one has a $\mathbf{P}^9$-bundle

$$\pi : \mathbb{P} \to \mathbf{P}^5 \quad (^{16})$$

with fibre at α the linear system of Nikulin surfaces $|\mathcal{I}_{A/T}(2)|$.

[16] onto a non empty open set of

- With some more elaboration:
  - The natural map $\mathbb{P}/Aut\ B \to \mathcal{F}_8^N$ is birational.
  - $\mathbb{P}/Aut\ B$ is birational to $\mathbf{P}^9 \times \Sigma$.
- We have sketched the proof that

Theorem

The moduli space of genus 8 Nikulin surfaces is rational.

- The Mukai construction for $\overline{S}$: Let
$$f : T \to \mathbf{P}^9,$$
defined by $|\mathcal{I}_A(2)|$.

- Let $\overline{T}$ be the birational image of f, then
$$\overline{T} = \mathbf{P}^9 \cap G(1,5).$$

- f contracts S_A to a copy of E_A spanning a hyperplane and
$$Sing\ \overline{T} = E_A.$$

Thanks for the attention!